A PRODUCT IDENTITY FOR SEQUENCES DEFINED BY $W_{n+2} = dW_{n+1} - cW_n$

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1. INTRODUCTION

Let W_0, W_1, $c \neq 0$, and $d \neq 0$ be arbitrary real numbers, and define

(1.1) $$W_{n+2} = dW_{n+1} - cW_n, \qquad d^2 - 4c \neq 0, \qquad (n = 0, 1, \cdots),$$

(1.2) $$Z_n = (a^n - b^n)/(a - b) \qquad (n = 0, 1, \cdots),$$

(1.3) $$V_n = a^n + b^n \qquad (n = 0, 1, \cdots),$$

where $a \neq b$ are the roots of $y^2 - dy + c = 0$. We shall define

(1.4) $$W_{-n} = (W_0V_n - W_n)/c^n \qquad (n = 0, 1, \cdots).$$

If $W_0 = 0$ and $W_1 = 1$, then $W_n \equiv Z_n$, $n = 0, 1, \cdots$; and if $W_0 = 2$ and $W_1 = d$, then $W_n \equiv V_n$, $n = 0, 1, \cdots$. The phrase, Lucas functions (of n) is often applied to Z_n and V_n.

It should be noted that

(1.5) $$W_n = W_0Z_{n+1} + (W_1 - dW_0)Z_n \qquad (n = 0, 1, \cdots);$$

and we shall refer to Z_n, $n = 0, 1, \cdots$, as the fundamental solution of (1.1). Let W_n^* be a second, general solution of (1.1) with initial values W_0^* and W_1^*. Since W_n^* also satisfies (1.5), we now see that the product sequence, $W_nW_n^*$, can be represented as a linear combination of Z_{n+1}^2, Z_mZ_{n+1}, and Z_n^2. We observe that

(1.6) $$W_nW_n = C_1a^{2n} + C_2b^{2n} + C_3c^n \qquad (n = 0, 1, \cdots),$$

where C_i, $i = 1, 2, 3$, are arbitrary constants, is the general solution of a third-order linear difference equation whose characteristic equation is

(1.7) $$(x - c)(x^2 - V_2x + c^2) = 0.$$

If the initial conditions of W_n and W_n^* are chosen such that $C_3 \equiv 0$, then $W_nW_n^*$ is also a solution of a second-order linear difference equation, and its representation is of interest.

2. STATEMENT OF RESULTS

Theorem 1. Let W_n and W_n^*, $n = 0, 1, \cdots$, be solutions of (1.1). Then (see (1.6))

$$W_2W_2^* - V_2W_1W_1^* + c^2W_0W_0^* = 0 \tag{2.1}$$

is a necessary and sufficient condition that $C_3 \equiv 0$. If $C_3 \equiv 0$, then

$$W_nW_n^* = ((W_1W_1^* - (d^2 - c)W_0W_0^*)/d)Z_{2n} + W_0W_0^*Z_{2n+1}\,; \tag{2.2}$$

and if $P_n \equiv W_nW_n^*$, then

$$P_{n+2} - V_2P_{n+1} + c^2P_n = 0 \qquad (n = 0, 1, \cdots)\,, \tag{2.3}$$

and

$$\frac{P_0 + (P_1 - V_2P_0)x}{1 - V_2x + c^2x^2} = \sum_{n=0}^{\infty} P_n x^n, \qquad (V_2 = d^2 - 2c)\,. \tag{2.4}$$

Corollary 1. If $d = -c = 1$, then $W_n \equiv H_n$, where H_n is the generalized Fibonacci number. Since $V_2 = 3$ and $Z_n \equiv F_n$, the ordinary Fibonacci number, we obtain from (2.2)

$$\begin{aligned} H_nH_n^* &= (H_1H_1^* - 2H_0H_0^*)F_{2n} + H_0H_0^*F_{2n+1} \\ &= H_1H_1^*F_{2n} - H_0H_0^*F_{2n-2} \end{aligned}\,, \tag{2.5}$$

(since $F_{2n+1} = 2F_{2n} - F_{2n-2}$), where (see (2.1))

$$H_2H_2^* - 3H_1H_1^* + H_0H_0^* = 0\,. \tag{2.6}$$

If $H_n^* = H_{n-1} + H_{n+1} \equiv G_n$, $n = 0, 1, \cdots$, then (2.6) is satisfied and thus (2.5) gives

$$H_nG_n = H_1G_1F_{2n} - H_0G_0F_{2n-2} \qquad (n = 0, 1, \cdots)\,; \tag{2.7}$$

and from (2.4), we obtain

$$\frac{H_0G_0 + (H_1G_1 - 3H_0G_0)x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} H_nG_nx^n\,. \tag{2.8}$$

Remarks. Our special result (2.7) solves completely the problem posed by Brother U. Alfred [1], where (2,9), for example, must stand for (H_0, H_1), and not, as incorrectly indicated (H_1, H_2). If $H_n \equiv F_n$, then $G_n \equiv L_n$, and (2.7) reduces to the well-known identity, $F_nL_n = F_{2n}$; and (2.8) gives

$$\frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n} x^n .$$

3. PROOF OF THEOREM 1

For $n = 0, 1,$ and 2, Eq. (1.6) gives a linear system of three equations for the three unknowns C_1, C_2, and C_3. We readily find that $C_3 = N/D$, where $D = cd(a - b)^3 \neq 0$ is the determinant of the system

$$W_0W_0^* = C_1 + C_2 + C_3 \tag{3.1}$$

$$W_1W_1^* = a^2C_1 + b^2C_2 + cC_3 \tag{3.2}$$

$$W_2W_2^* = a^4C_1 + b^4C_2 + c^2C_3 \tag{3.3}$$

and

$$N = \begin{vmatrix} 1 & 1 & W_0W_0^* \\ a^2 & b^2 & W_1W_1^* \\ a^4 & b^4 & W_2W_2^* \end{vmatrix} .$$

If we set $N = 0$, we obtain the necessary condition (2.1) for $C_3 = 0$.

For the sufficiency proof, we assume that (2.1) is true. If we multiply both sides of (3.1) by c^2 and both sides of (3.2) by $-V_2$, then the addition of the resulting equations to (3.3) gives, using (2.1),

$$0 = (c^2 - a^2V_2 + a^4)C_1 + (c^2 - b^2V_2 + b^4)C_2 + (c^2 - cV_2 + c^2)C_3 . \tag{3.4}$$

Since $c = ab$ and $V_2 = a^2 + b^2$, we obtain from (3.4)

$$0 = -ab(a - b)^2C_3 .$$

Since $a \neq b \neq 0$, we must have $C_3 = 0$.

If $C_3 \equiv 0$, then (see (1.6))

$$P_n \equiv W_nW_n^* = C_1a^{2n} + C_2b^{2n}, \qquad n = 0, 1, \cdots .$$

Since $P_0 = C_1 + C_2$, we obtain, respectively, noting (1.2),

$$P_n = C_2(b - a)Z_{2n} + P_0a^{2n} \qquad (n = 0, 1, \cdots), \tag{3.5}$$

$$P_n = C_1(a - b)Z_{2n} + P_0b^{2n} \qquad (n = 0, 1, \cdots). \tag{3.6}$$

Evaluating C_2 in (3.5) (for $n = 1$) and C_1 in (3.6) (for $n = 1$), we obtain, respectively, after simplification,

$$P_n = [(P_1 - a^2P_0)/d]Z_{2n} + P_0a^{2n} \qquad (n = 0, 1, \cdots), \tag{3.7}$$

$$P_n = [(P_1 - b^2P_0)/d]Z_{2n} + P_0b^{2n} \qquad (n = 0, 1, \cdots). \tag{3.8}$$

Addition of (3.7) and (3.8) gives

$$2P_n = [(2P_1 - V_2P_0)/d]Z_{2n} + P_0V_{2n} \qquad (n = 0, 1, \cdots). \tag{3.9}$$

Since (see (1.5)) $V_{2n} = 2Z_{2n+1} - dZ_{2n}$, we obtain from (3.9)

$$2dP_n = (2P_1 - P_0(V_2 + d^2))Z_{2n} + 2dP_0Z_{2n+1}. \tag{3.10}$$

Noting that $V_2 + d^2 = 2d^2 - 2c$, we obtain from (3.10),

$$P_n = [(P_1 - P_0(d^2 - c))/d]Z_{2n} + P_0Z_{2n+1}. \tag{3.11}$$

Since $P_n \equiv W_nW_n^*$, Eq. (3.11) reduces to (2.2).

If we set $(E^2 - V_2E + c^2)W_nW_n^* = Q_n$, where $E^mA_n = A_{n+m}$, then (1.7) becomes

$$(E - c)Q_n = 0. \tag{3.12}$$

The solution to (3.12) is

$$Q_n = Kc^n \qquad (K, \text{ a constant}). \tag{3.13}$$

But $K = Q_0$, and so (3.13) reads

$$W_{n+2}W_{n+2}^* - V_2W_{n+1}W_{n+1}^* + c^2W_nW_n^* = Q_0c^n, \tag{3.14}$$

where

$$Q_0 = W_2W_2^* - V_2W_1W_1^* + c^2W_0W_0^*.$$

If (2.1) is true, then $Q_0 = 0$, and $P_n \equiv W_nW_n^*$ satisfies (2.3); and (2.4) follows readily from (2.3).

4. COMMENTS

If $W_n^* = W_{n-1} - (1/c)W_{n+1}$ in Theorem 1, then (2.1) is satisfied. For example, if $W_{n+2} = 2W_{n+1} + W_n$, then $\{Z_n\}_0^\infty = \{0, 1, 2, 5, 12, \cdots\}$, where Z_n is Pell's sequence. If we choose

$$\{W_n\}_0^\infty = \{2, 3, 8, 19, \cdots\}$$

and set

$$W_n^* = W_{n-1} + W_{n+1},$$

then

$$\{W_n\}_0^\infty = \{2, 10, 22, \cdots\};$$

and since $d = 2$ and $c = -1$, we obtain from (2.2) in Theorem 1

$$W_nW_n^* = 5Z_{2n} + 4Z_{2n+1} \qquad (n = 0, 1, \cdots), \tag{4.1}$$

where Z_n is Pell's sequence.

Using results of the author [2, p. 242], it seems reasonable that the conclusions of Theorem 1 may be extended (properly interpreted) to p products of solutions of (1.1), where $p = 2, 4, 6, \cdots$. For example, if $P_n = W_nW_n^*W_n^{**}W_n^{***}$, where W_n, W_n^*, W_n^{**}, and W_n^{***} are independent solutions of (1.1), then P_n satisfies a fifth-order linear difference equation (see [2, (2.2), p. 242] whose characteristic equation is

$$(x - c^2)\prod_{j=0}^{1}(x^2 - c^jV_{4-2j}x + c^4) = 0. \tag{4.2}$$

Since

$$P_n = C_1a^{4n} + C_2(a^3b)^n + C_3c^{2n} + C_4(ab^3)^n + C_5b^{4n},$$

we believe that $C_3 \equiv 0$ if and only if

$$\left[\prod_{j=0}^{1}(E^2 - c^jV_{4-2j}E + c^4)\right]P_0 = 0. \tag{4.3}$$

However, the representation of P_n under (4.3) is another matter.

For the case $d^2 = 4c$, $d \neq 0$, it appears that (2.2) of Theorem 1 holds under (2.1). Moreover, if $2W_1 = dW_0$, then (2.1) holds for any arbitrary sequence W_n^*. Since $a = b$, we have $Z_n = na^{n-1}$, $n = 0, 1, \cdots$, in (2.2).

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$$B(t,t) = \sum_{k=0}^{t-1} \binom{t-1}{k} \frac{(-1)^k}{t+k} .$$

Hence $y \pmod{10^{tn}}$, defined by (8), with coefficients given by (10) and (12), is an automorphic number of tn places. By replacing $k - t$ by k, we get the representation (1). Further, by using identity (5),

$$y = t\binom{2t-1}{t} x^t \sum_{k=0}^{t-1} \frac{(-x)^k}{t+k} \binom{t-1}{k},$$

where

$$\frac{1}{x^t}\int_0^x u^{t-1}(1-u)^{t-1}du = \int_0^1 v^{t-1}(1-xv)^{t-1}dv$$

$$= \sum_{k=0}^{t-1} \binom{t-1}{k} \frac{(-x)^k}{t+k} ,$$

by expanding $(1 - xv)^{t-1}$ and integrating term-by-term. This result yields the representation (2).

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