# ENUMERATION OF 3×3 ARRAYS

L. CARLITZ*
Duke University, Durham, North Carolina

1. Let

$$A = (a_{ij}) \qquad (i,\ j = 1,\ 2,\ 3) \tag{1.1}$$

denote an array of non-negative integers. Let $H(r)$ denote the number of arrays (1.1) such that

$$\sum_{j=1}^{n} a_{ij} = r = \sum_{j=1}^{n} a_{ji} \qquad (i = 1,\ 2,\ 3)\ . \tag{1.2}$$

MacMahon [2, p. 161] has proved that

$$\sum_{r=0}^{\infty} H(r)x^r = \frac{1 - x^3}{(1 - x)^6} = \frac{1 + x + x^2}{(1 - x)^5}\ . \tag{1.3}$$

This result has recently been rediscovered by Anand, Demir and Gupta [1].

Let $H(r,t)$ denote the number of arrays (1.1) that satisfy (1.2) and also

$$\sum_{i=1}^{3} a_{ii} = t \tag{1.4}$$

and let $H(r,s,t)$ denote the number of arrays (1.1) that satisfy (1.2), (1.4) and

$$\sum_{i=1}^{3} a_{i,4-i} = s\ . \tag{1.5}$$

MacMahon [2, pp. 162-163] has proved that

$$\sum_{r=0}^{\infty} H(r,r)x^r = \frac{1 - x^6}{(1 - x)^3(1 - x^3)^2} = \frac{1 + x^3}{(1 - x)^3(1 - x^3)} \tag{1.6}$$

* Supported in part by NSF Grant GP-7855.

and

$$\sum_{r=0}^{\infty} H(r,r,r)x^r = \frac{(1-x^6)^2}{(1-x^3)^5} = \frac{(1+x^3)^2}{(1-x^3)^3} \tag{1.7}$$

In the present paper we show first that if

$$\overline{H}(r;\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4) = \sum_{\substack{a+b\le r\\ c+d\le r\\ a+c\le r\\ b+d\le r\\ a+b+c+d\ge r}} \lambda_1^a\lambda_2^b\lambda_3^c\lambda_4^d \tag{1.8}$$

then

$$\sum_{r=0}^{\infty} \overline{H}(r;\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4)x^r$$

$$= \frac{1-\lambda_1\lambda_2\lambda_3\lambda_4x^3}{(1-\lambda_1x)(1-\lambda_2x)(1-\lambda_3x)(1-\lambda_4x)(1-\lambda_1\lambda_4x)(1-\lambda_2\lambda_3x)} \tag{1.9}$$

We show next that (1.9) implies

$$\sum_{r,s,t=0}^{\infty} H(r,s,t)x^r y^s z^t$$

$$= \frac{1-x^3y^3z^3}{(1-xy)^2(1-xz)^2(1-xy^3z)(1-xyz^3)} \tag{1.10}$$

This in turn implies

$$\sum_{r,t=0}^{\infty} H(r,t)\, x^r z^t = \frac{1-x^3z^3}{(1-x)^2(1-xz)^3(1-xz^3)} \tag{1.11}$$

which we show implies (1.6).

In the next place we prove

$$\sum_{r,t=0}^{\infty} H(r,t,t)\, x^r z^t$$

$$= \frac{1+x^2z+4x^3z^3-4x^5z^4-x^6z^6-x^8z^7}{(1-x^2z^4)(1-x^3z^3)(1-x^2z)^3} \tag{1.12}$$

which we show implies (1.7). We also give a combinatorial proof of (1.7).

Finally, if

$$K(s,t) = \begin{cases} H(r,s,t) & (s + t = 2r) \\ 0 & (s + t \text{ odd}) \end{cases}, \tag{1.13}$$

we show that

$$\sum_{s,t=0}^{\infty} K(s,t)\, y^s z^t \tag{1.14}$$

$$= \frac{(1 + y^3z^3)[1 + 4y^3z^3 + y^6z^6 + 4y^2z^2(y^2 + z^2) + yz(y^4 + z^4)]}{(1 - y^5 z)^2(1 - y z^5)^2} .$$

Moreover (1.14) contains (1.7).

2. Proof of (1.9). It follows from (1.8) that

$$\overline{H}(r;\ \lambda_1, \lambda_2, \lambda_3, \lambda_4) = S_1(r) - S_2(r) , \tag{2.1}$$

where

$$S_1(r) = \sum_{\substack{a+b\le r\\ c+d\le r\\ a+c\le r\\ b+d\le r}} \lambda_1^a\lambda_2^b\lambda_3^c\lambda_4^d \tag{2.2}$$

and

$$S_2(r) = \sum_{a+b+c+d<r} \lambda_1^a\lambda_2^b\lambda_3^c\lambda_4^d . \tag{2.3}$$

Then by (2.2)

$$S_1(r) = \sum_{b,c\le r} \lambda_2^b\lambda_3^c \sum_{\substack{a\le r-b\\ a\le r-c}} \lambda_1^a \sum_{\substack{d\le r-b\\ d\le r-c}} \lambda_4^d$$

$$= \sum_{b,c\le r} \lambda_2^{r-b}\lambda_3^{r-c} \sum_{\substack{a\le b\\ a\le c}} \lambda_1^a \sum_{\substack{d\le b\\ d\le c}} \lambda_4^d$$

$$= \sum_{b\le c\le r} \lambda_2^{r-b}\lambda_3^{r-c} \sum_{a\le b} \lambda_1^a \sum_{d\le b} \lambda_4^d$$

$$+ \sum_{c\le b\le r} \lambda_2^{r-b}\lambda_3^{r-c} \sum_{a\le c} \lambda_1^a \sum_{d\le c} \lambda_4^d$$

$$- \sum_{b \le r} \lambda_2^{r-b} \lambda_3^{r-b} \sum_{a \le b} \lambda_1^a \sum_{d \le b} \lambda_4^d$$

$$= \sum_{b \le r} \lambda_2^{r-b} \frac{1 - \lambda_3^{r-b+1}}{1 - \lambda_3} \frac{1 - \lambda_1^{b+1}}{1 - \lambda_1} \frac{1 - \lambda_4^{b+1}}{1 - \lambda_4}$$

$$+ \sum_{c \le r} \lambda_3^{r-c} \frac{1 - \lambda_2^{r-c+1}}{1 - \lambda_2} \frac{1 - \lambda_1^{c+1}}{1 - \lambda_1} \frac{1 - \lambda_4^{c+1}}{1 - \lambda_4}$$

$$- \sum_{b \le r} \lambda_2^{r-b} \lambda_3^{r-b} \frac{1 - \lambda_1^{b+1}}{1 - \lambda_1} \frac{1 - \lambda_4^{b+1}}{1 - \lambda_4} .$$

It follows that

$$\sum_{r=0}^{\infty} S_1(r)\, x^r = \sum_{b=0}^{\infty} \frac{1 - \lambda_1^{b+1}}{1 - \lambda_1} \frac{1 - \lambda_4^{b+1}}{1 - \lambda_4} x^b \sum_{r=0}^{\infty} \lambda_2^r \frac{1 - \lambda_3^{r+1}}{1 - \lambda_3} x^r$$

$$+ \sum_{c=0}^{\infty} \frac{1 - \lambda_1^{c+1}}{1 - \lambda_1} \frac{1 - \lambda_4^{c+1}}{1 - \lambda_4} x^c \sum_{r=0}^{\infty} \lambda_3^r \frac{1 - \lambda_2^{r+1}}{1 - \lambda_2} x^r$$

$$- \sum_{b=0}^{\infty} \frac{1 - \lambda_1^{b+1}}{1 - \lambda_1} \frac{1 - \lambda_4^{b+1}}{1 - \lambda_4} x^b \sum_{r=0}^{\infty} \lambda_2^r \lambda_3^r x^r .$$

Carrying out the summations and simplifying, we get

$$(2.4) \qquad \sum_{r=0}^{\infty} S_1(r) x^r = \frac{(1 - \lambda_1\lambda_4 x^2)(1 - \lambda_2\lambda_3 x^2)}{(1 - x)(1 - \lambda_1 x)(1 - \lambda_2 x)(1 - \lambda_3 x)(1 - \lambda_4 x)(1 - \lambda_1\lambda_4 x)(1 - \lambda_2\lambda_3 x)} .$$

Similarly we find that

$$(2.5) \qquad \sum_{r=0}^{\infty} S_2(r)\, x^r = \frac{x}{(1 - x)(1 - \lambda_1 x)(1 - \lambda_2 x)(1 - \lambda_3 x)(1 - \lambda_4 x)} .$$

Since, by (2.1)

$$\sum_{r=0}^{\infty} \overline{H}(r;\ \lambda_1, \lambda_2, \lambda_3, \lambda_4)x^r = \sum_{r=0}^{\infty} S_1(r)x^r - \sum_{r=0}^{\infty} S_2(r)x^r ,$$

it is easily verified that we get (1.9).

3. Proof of (1.10) and (1.6). Consider the array

$$\text{(3.1)} \qquad \begin{array}{|ccc|} a & b & r-a-b \\ c & d & r-c-d \\ r-a-c & r-b-d & k \end{array} \; .$$

If

$$\text{(3.2)} \qquad a + b + c + d = k + r$$

then clearly all row and column sums of (3.1) equal $r$. It follows that

$$\overline{H}(r;\ 1, 1, 1, 1) = H(r) .$$

Let

$$\text{(3.3)} \qquad a + d + k = t$$

and

$$\text{(3.4)} \qquad 2r - 2a - b - c + d = s .$$

It follows from (3.2) and (3.3) that

$$\text{(3.5)} \qquad t + r = 2a + b + c + 2d .$$

In (1.9) take

$$\text{(3.6)} \qquad \lambda_1 = y^{-2}z^2, \qquad \lambda_2 = \lambda_3 = y^{-1}z, \qquad \lambda_4 = yz^2$$

and replace $x$ by $xy^2z^{-1}$. The left member of (1.9) becomes

$$\sum_{r=0}^{\infty} \overline{H}(r;\ y^{-2}z^2,\ y^{-1}z,\ yz^2)(xy^2z^{-1})^r$$

$$= \sum_{r=0}^{\infty} x^r \sum_{\substack{a+b\le r\\ c+d\le r\\ a+c\le r\\ a+b+c+d\ge r}} y^{2r-2a-b-c+d}\, z^{2a+b+c+2d-r}$$

$$= \sum_{r,s,t=0}^{\infty} H(r,s,t)\, x^r y^s z^t ,$$

where $H(r,s,t)$ is the number of arrays (3.1) that satisfy (3.3) and (3.4). We have therefore the generating function

$$(3.7)\qquad \sum_{r,s,t=0}^{\infty} H(r,s,t)\, x^r y^s z^t = \frac{1 - x^3 y^3 z^3}{(1 - xy)^2(1 - xz)^2(1 - xy^3z)(1 - xyz^3)}$$

To obtain a generating function for $H(r,t)$, the number of arrays (3.1) that satisfy (3.3), we take $y = 1$. Thus

$$(3.8)\qquad \sum_{r,t=0}^{\infty} H(r,t)\, x^r z^t = \frac{1 - x^3 z^3}{(1 - x)^2(1 - xz)^3(1 - xz^3)} .$$

We shall now show that (3.8) implies (1.6). Since

$$(1 - x)^{-2}(1 - xz^3)^{-1} = \sum_{a,b=0}^{\infty} (a + 1)x^{a+b} z^{3b} ,$$

it follows that the terms in which the exponents of $x$ and $z$ are equal contribute

$$\sum_{b=0}^{\infty} (2b + 1)x^{3b} z^{3b} = \frac{1 + x^3z^3}{(1 - x^3 z^3)^2} .$$

Therefore

$$\sum_{r=0}^{\infty} H(r,r)x^r = \frac{1 - x^3}{(1 - x)^3}\ \frac{1 + x^3}{(1 - x^3)^2} = \frac{1 + x^3}{(1 - x)^3(1 - x^3)} .$$

4. Proof of (1.12) and (1.7). Returning to (3.7), we shall now obtain a generating function for $H(r,t,t)$. We have

$$(1 - xy)^{-2}(1 - xz)^{-2}(1 - xy^3z)^{-1}(1 - xyz^3)^{-1}$$

$$= \sum_{a,b,c,d=0}^{\infty} (a + 1)(b + 1)x^{a+b+c+d}\, y^{a+3c+d}\, z^{b+c+3d} .$$

For those terms in which $y$ and $z$ have equal exponents

$$a + 3c + d = b + c + 3d ,$$

so that

$$a + 2c = b + 2d .$$

We accordingly get

$$2\sum_{c\le d}\sum_{b}(b-2c+2d+1)(b+1)x^{2b-c+3d}(yz)^{b+c+3d}-\sum_{a,c}(a+1)^2x^{2a+2c}(yz)^{a+4c}$$

$$=2\sum_{b,c,d}(b+2d+1)(b+1)x^{2b+2c+d}(yz)^{b+4c+3d}-\sum_{a,c}(a+1)^2x^{2a+2c}(yz)^{a+4c}$$

$$=\frac{2}{1-x^2(yz)^4}\sum_{b,d}(b+2d+1)(b+1)x^{2b+3d}(yz)^{b+3d}-\frac{1}{1-x^2(yz)^4}\sum_{a}(a+1)^2x^{2a}(yz)^a.$$

Carrying out the indicated summations, we get

$$\frac{1}{1-x^2(yz)^4}\left\{\frac{2}{1-x^3(yz)^3}\,\frac{1+x^2yz}{(1-x^2yz)^2}+\frac{4(xyz)^3}{(1-x^2yz)(1-x^3)(yz^3)^2}-\frac{1+x^2yz}{(1-x^2yz)^3}\right\}$$

which reduces to

$$\frac{1+x^2yz+4x^3(yz)^3-4x^5(yz)^4-x^6(yz)^6-x^8(yz)^7}{(1-x^2(yz)^4)(1-x^3(yz)^3)^2(1-x^2yz)^3}.$$

It follows therefore that

$$\sum_{r,t=0}^{\infty}H(r,t,t)x^r z^t$$

(4.1)

$$=\frac{1+x^2z+4x^3z^3-4x^5z^4-x^6z^6-x^8z^7}{(1-x^2z^4)(1-x^3z^3)(1-x^2z)^3}.$$

To get a generating function for $H(r,r,r)$ we observe that the right member of (4.1) is equal to

$$\frac{(1+x^2z)(1+x^3z^3)}{(1-x^2z^4)(1-x^2z)^3}+\frac{4x^3z^3}{(1-x^2z^4)(1-x^3z^3)(1-x^2z)^2}. \tag{4.2}$$

The first fraction

$$=(1+x^3z^3)\sum_{a,b}(a+1)^2x^{2a+2b}z^{a+4b}$$

which will contribute

$$(1+x^3z^3)\sum_{b}(2b+1)^2(xz)^{6b}=\frac{(1+x^3z^3)(1+6x^6+x^{12}z^{12})}{(1-x^6z^6)^3}.$$

The second fraction in (4.2)

$$= \frac{4x^3z^3}{1 - x^3z^3} \sum_{a,b} (a + 1)\, x^{2a+2b}\, z^{a+4b}$$

which will contribute

$$\frac{4x^3z^3}{1 - x^3z^3} \sum_{b} (2b + 1)(xz)^{6b} = \frac{4x^3z^3}{1 - x^3z^3}\, \frac{1 + x^6z^6}{(1 - x^6z^6)^2} .$$

The total contribution is evidently

$$\frac{(1 + x^3z^3)^4}{(1 - x^3z^3)(1 - x^6z^6)^2} = \frac{(1 + x^3z^3)^2}{(1 - x^3z^3)^3} .$$

We have therefore

$$\sum_{r=0}^{\infty} H(r, r, r)x^r = \frac{(1 + x^3)^2}{(1 - x^3)^3} . \tag{4.3}$$

As noted by MacMahon, Eq. (4.3) is equivalent to

$$H(3m, 3m, 3m) = m^2 + (m + 1)^2 . \tag{4.4}$$

We shall now give a combinatorial proof of (4.4). With the notation (3.1) it is clear that $H(r, r, r)$ is equal to the number of solutions of the following system

$$\begin{cases} a + b + c + d = k + r \\ k + a + d = r \\ 2a + b + c - d = r \\ a + b \le r, \quad c + d \le r \\ a + c \le r, \quad b + d \le r \end{cases} . \tag{4.5}$$

It follows that $3d = r$. Thus, for $r = 3m$, Eq. (4.5) reduces to

$$\begin{cases} 2a + b + c = 4m \\ a + b \le 3m \\ a + c \le 3m \\ b \le 2m, \quad c \le 2m \end{cases} . \tag{4.6}$$

For $0 \le a \le m$, Eq. (4.6) implies

$$b \ge 2m - 2a, \qquad c \ge 2m - 2a .$$

Hence

$$H(3m,3m,3m) = \sum_{a=0}^{m-1} \sum_{2m-2a\le b\le 2m} 1 + \sum_{a=m}^{2m} \sum_{b+c=4m-2a} 1$$

$$= \sum_{a=0}^{m-1} (2a+1) + \sum_{a=m}^{2m} (4m - 2a + 1)$$

$$= m^2 + \sum_{a=0}^{m} (2a+1)$$

$$= m^2 + (m+1)^2 \quad .$$

5. Proof of (1.14). Returning to (1.10) we replace $x$ by $x^2$, $y$ by $x^{-1}y$ and $z$ by $x^{-1}z$. If $K(s,t)$ is defined by (1.13) it is clear that

$$\sum_{s,t=0}^{\infty} K(s,t)\, y^s z^t$$

is equal to the sum of the terms in

$$\frac{1 - y^3z^3}{(1-xy)^2(1-xz)^2(1-x^{-2}y^3z)(1-x^{-2}yz^3)} \tag{5.1}$$

that are independent of $x$. Expanding (5.1), this sum is seen to be

$$(1 - y^3z^3) \sum (a+1)(b+1) y^{a+3c+d} z^{b+c+3d} \quad ,$$

where the summation is over all non-negative $a, b, c, d$ such that $a + b = 2c + 2d$. This gives

$$(1 - y^3z^3) \sum_{a,b=o}^{\infty} (2a+1)(2b+1) y^{2a} z^{2b} (yz)^{a+b} \sum_{c+d=a+b} y^{2c} z^{2d}$$

$$+ (1 - y^3z^3) \sum_{a,b=o}^{\infty} (2a+2)(2b+2) y^{2a+1} z^{2b+1} (yz)^{a+b+1} \sum_{c+d=a+b+1} y^{2c} z^{2d} \; .$$

Carrying out the indicated summations, we get

$$\frac{1-y^3z^2}{y^2-z^2}\left\{y^2\,\frac{1+y^5z}{(1-y^5z)^2}\,\frac{1+y^3z^3}{(1-y^3z^3)^2} - z^2\,\frac{1+y^3z^3}{(1-y^3z^3)^2}\,\frac{1+yz^5}{(1-yz^5)^2}\right.$$
$$\left. + 4y^4\,\frac{y^2z^2}{(1-y^5z)^2(1-y^3z^3)^2} - 4z^4\,\frac{y^2z^2}{(1-y^3z^3)^2(1-yz^5)^2}\right\} .$$

A little manipulation gives

$$\frac{(1+y^3z^3)\,[1+4y^3z^3+y^6z^6+4y^2z^2(y^2+z^2)+yz(y^4+z^4)]}{(1-y^5z)^2(1-yz^5)^2} .$$

This completes the proof of (1.14).

To show that (1.14) contains (1.7), we take

$$(1-y^5z)^{-2}(1-yz^5)^{-2} = \sum_{a,b=o}^{\infty}(a+1)(b+1)y^{5a+b}z^{a+5b} .$$

Since

$$1+4y^3z^3+y^6z^6+4y^2z^2(y^2+z^2)+yz(y^4+z^4)$$
$$= 2(1+y^3z^3)^2 - (1-y^5z)(1-yz^5) + 4y^2z^2(y^2+z^2) ,$$

it follows that

$$\sum_{s=o}^{\infty} H(s,s,s)z^s = \sum_{s=o}^{\infty} K(s,s)z^s$$
$$= 2(1+z^3)^3\sum_{a=o}^{\infty}(a+1)^2z^{6a} - (1+z^3)\sum_{a=o}^{\infty}z^{6a}$$
$$= 2(1+z^3)^3\,\frac{1+z^6}{(1-z^6)^3} - \frac{1+z^3}{1-z^6}$$
$$= \frac{2(1+z^6)}{(1-z^3)^3} - \frac{1}{1-z^3} = \frac{(1+z^3)^2}{(1-z^3)^3} .$$

## REFERENCES

1. H. Anand, V. C. Dumir, and H. Gupta, "A Combinatorial Distribution Problem," Duke Mathematical Journal, Vol. 33 (1966), pp. 757-769.
2. P. A. MacMahon, Combinatory Analysis, Vol. 2, Cambridge, 1916.