FIBONACCI AND LUCAS TRIANGLES

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1. INTRODUCTION

We first define four sequences of polynomials and lay out two fundamental identities. Let

(a) $$f_0(x) = 0, \quad f_1(x) = 1, \quad \text{and} \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x) .$$

These are the Fibonacci polynomials, and $f_n(1) = F_n$. Let

(b) $$L_0(x) = 2, \quad L_1(x) = x, \quad \text{and} \quad L_{n+2}(x) = xL_{n+1}(x) + L_n(x) ,$$

which are the Lucas polynomials. It is easy to show that

$$L_n(x) = f_{n+1}(x) + f_{n-1}(x), \quad L_n^2(x) - (x^2+4)f_n^2(x) = (-1)^n 4, \quad \text{and} \quad L_n(1) = L_n .$$

These are two well known polynomial sequences which have been much discussed in these pages. Both enjoy Binet forms. Let $\lambda^2 - x\lambda - 1 = 0$ have roots

$$\lambda_1 = \frac{x + \sqrt{x^2+4}}{2} \quad \text{and} \quad \lambda_2 = \frac{x - \sqrt{x^2+4}}{2} .$$

Then

$$f_n(x) = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \quad \text{and} \quad L_n(x) = \lambda_1^n + \lambda_2^n ,$$

$$\lambda_1^n = \frac{L_n(x) + \sqrt{x^2+4}\, f_n(x)}{2} \quad \text{and} \quad \lambda_2^n = \frac{L_n(x) - \sqrt{x^2+4}\, f_n(x)}{2} .$$

Next we introduce two polynomial sequences closely related to the Chebysheff polynomials of the first and second kind which were introduced in [2]. Let

$$g_0(x) = 0, \quad g_1(x) = 1, \quad \text{and} \quad g_{n+2}(x) = xg_{n+1}(x) - g_n(x) ,$$

$$h_0(x) = 2, \quad h_1(x) = x, \quad \text{and} \quad h_{n+2}(x) = xh_{n+1}(x) - h_n(x) .$$

It is easy to establish

$$h_n^2(x) - (x^2 - 4)g_n^2(x) = 4 \quad \text{and} \quad h_n(x) = g_{n+1}(x) + g_{n-1}(x),$$

and if $\lambda^2 - x\lambda + 1 = 0$ has roots

$$\lambda_1^*(x) = \frac{x + \sqrt{x^2 - 4}}{2} \quad \text{and} \quad \lambda_2^*(x) = \frac{x - \sqrt{x^2 - 4}}{2},$$

then, for $x \neq \pm 2$,

$$g_n(x) = \frac{\lambda_1^{*n}(x) - \lambda_2^{*n}(x)}{\lambda_1^*(x) - \lambda_2^*(x)} \quad \text{and} \quad h_n(x) = \lambda_1^{*n}(x) + \lambda_2^{*n}(x),$$

while $g_n(2) = n$ and $g_n(-2) = -n$, $n = 0, 1, 2, \cdots$. As with Fibonacci polynomials, $g_n(x)$ have their coefficients lying along the rising diagonals of Pascal's triangle.

2. SUBSTITUTIONS INTO POLYNOMIAL SEQUENCES

Consider

$$\lambda_1(x) = \frac{x + \sqrt{x^2 + 4}}{2}$$

for x replaced by $L_{2n+1}(x)$. From $L_n^2(x) - (x^2 + 4)f_n^2(x) = 4(-1)^n$ we see that

$$\lambda_1(L_{2n+1}(x)) = \frac{L_{2n+1}(x) + \sqrt{x^2 + 4}\, f_{2n+1}(x)}{2}$$

which from

$$\lambda_1^n(x) = [L_n(x) + \sqrt{x^2 + 4}\, f_n(x)]/2$$

becomes

$$\lambda_1(L_{2n+1}(x)) = \lambda_1^{2n+1}(x).$$

Similarly,

$$\lambda_2(L_{2n+1}(x)) = \lambda_2^{2n+1}(x).$$

Now let us look at

$$f_m(L_{2n+1}(x)) = \frac{\lambda_1^m(L_{2n+1}(x)) - \lambda_2^m(L_{2n+1}(x))}{\lambda_1(L_{2n+1}(x)) - \lambda_2(L_{2n+1}(x))}$$

$$= \frac{\lambda_1^{m(2n+1)}(x) - \lambda_2^{m(2n+1)}(x)}{\lambda_1^{2n+1}(x) - \lambda_2^{2n+1}(x)} = \frac{f_{m(2n+1)}(x)}{f_{2n+1}(x)}$$

by dividing numerator and denominator by $\lambda_1(x) - \lambda_2(x)$ and using the Binet form for the Fibonacci polynomials. We note that since the coefficients of both polynomial sequences $f_n(x)$

and $L_n(x)$ are integers, then $f_m(L_{2n+1}(x))$ is a polynomial and $f_{m(2n+1)}(x)/f_{2n+1}(x)$ is a polynomial. Letting $x = 1$ shows that $F_{2n+1} \mid F_{m(2n+1)}$.

If instead we use $L_{2n}^2(x) - 4 = (x^2 + 4)f_{2n}^2(x)$, then

$$\lambda_1^*(x) = \frac{x + \sqrt{x^2 - 4}}{2}, \qquad \lambda_2^*(x) = \frac{x - \sqrt{x^2 - 4}}{2}$$

becomes

$$\lambda_1^*(L_{2n}(x)) = \frac{L_{2n}(x) + \sqrt{x^2 + 4}\, f_{2n}(x)}{2} = \lambda_1^{2n}(x)$$

$$\lambda_2^*(L_{2n}(x)) = \frac{L_{2n}(x) - \sqrt{x^2 + 4}\, f_{2n}(x)}{2} = \lambda_2^{2n}(x)$$

so that

$$g_m(x) = \frac{\lambda_1^{*m}(x) - \lambda_2^{*m}(x)}{\lambda_1^*(x) - \lambda_2^*(x)}$$

becomes, when x is replaced by $L_{2n}(x)$,

$$g_m(L_{2n}(x)) = \frac{\lambda_1^{*m}(L_{2n}(x)) - \lambda_2^{*m}(L_{2n}(x))}{\lambda_1^*(L_{2n}(x)) - \lambda_2^*(L_{2n}(x))}$$

$$= \frac{\lambda_1^{2mn}(x) - \lambda_2^{2mn}(x)}{\lambda_1^{2n} - \lambda_2^{2n}} = \frac{f_{2mn}(x)}{f_{2n}(x)}$$

as before using the Binet form for the Fibonacci polynomials. Again $g_m(L_{2n}(x))$ is a polynomial when $x = 1$, $F_{2n} \mid F_{2mn}$.

Summarizing,

(A) $$f_{(2n+1)m}(x) = f_{2n+1}(x)\, f_m(L_{2n+1}(x))$$

(B) $$f_{2nm}(x) = f_{2n}(x)\, g_m(L_{2n}(x)) .$$

Using the explicit formulas for the polynomials $f_n(x)$ and $g_n(x)$, we have

$$f_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n+1-k}{k} x^{n-2k}, \qquad g_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n+1-k}{k} (-1)^k x^{n-2k}$$

Then, we can combine (A) and (B) into one formula,

$$f_{nk}(x) = f_k(x) \sum_{j=0}^{[n/2]} \binom{n-1-j}{j} (-1)^{(k+1)(j+1)}\, L_k^{n-2j-1}(x) .$$

This justifies the formula reported by Brother Alfred in [3], Table 41, when $x = 3$, but, of course, holds for other x as well.

3. THE LUCAS TRIANGLE

The polynomial sequences $L_n(x)$ and $h_n(x)$ for each n have the same coefficients except those of $h_n(x)$ alternate in sign. If we call the coefficient array for the Lucas polynomials the Lucas Triangle, then we can get a result similar to that above as reported for Table 42 in [3]. See also [1], [4], [5], and [6]. First,

$$\lambda_1(L_{2n+1}(x)) = \lambda_1^{2n+1}(x) \qquad \text{and} \qquad \lambda_2(L_{2n+1}(x)) = \lambda_2^{2n+1}(x)$$

so that

$$L_m(L_{2n+1}(x)) = \lambda_1^{(2n+1)m}(x) + \lambda_2^{(2n+1)m}(x) = L_{m(2n+1)}(x) .$$

Next

$$\lambda_1^*(L_{2n}(x)) = \lambda_1^{2n}(x) \qquad \text{and} \qquad \lambda_2^*(L_{2n}(x)) = \lambda_2^{2n}(x)$$

so that

$$h_n(L_{2n}(x)) = \lambda_1^{2mn}(x) + \lambda_2^{2mn}(x) = L_{2mn}(x) .$$

This evidently establishes the counterpart for the Lucas polynomials.

We note in passing that $L_0(x) = 2$, $L_1(x) = x$, and from

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \qquad L_2(x) = x^2 + 2 .$$

Thus the $L_{2n+1}(x)$ are divisible by x. This also holds for $h_{2n+1}(x)$.

Thus,

$$L_{2m+1}(L_{2n+1}(x)) = L_{(2m+1)(2n+1)}(x)$$
$$h_{2m+1}(L_{2n}(x)) = L_{(2m+1)(2n)}(x)$$

implies that $L_p(x) \mid L_{(2m+1)p}(x)$. Similarly, $f_p(x) \mid f_{mp}(x)$. Setting $x = 1$ establishes $L_p \mid L_{(2m+1)p}$ and $F_p \mid F_{mp}$ for $m \geq 0$.

4. SOME OTHER RESULTS

Suppose

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x); \qquad f_0(x) = 0, \quad f_1(x) = 1 .$$

Next let $x = \alpha$, where $\alpha^2 = \alpha + 1$; then

$$f_n(\alpha) = \alpha P_n + Q_n .$$

Here we seek recurrences for the sequences P_n and Q_n. Thus

$$\alpha P_{n+2} + Q_{n+2} = \alpha(\alpha P_{n+1} + Q_{n+1}) + (\alpha P_n + Q_n)$$

and

$$P_{n+2} = P_{n+1} + P_n + Q_{n+1}$$

$$Q_{n+2} = P_{n+1} + Q_n$$

$$P_{n+1} = P_n + P_{n-1} + Q_n$$

$$P_{n+3} = P_{n+2} + P_{n+1} + Q_{n+2} .$$

Subtracting

$$P_{n+3} - P_{n+1} = P_{n+2} - P_n + P_{n+1} - P_{n-1} + Q_{n+2} - Q_n .$$

Thus

$$P_{n+3} = P_{n+2} + 3P_{n+1} - P_n - P_{n-1}$$

since

$$Q_{n+2} - Q_n = P_{n+1} ,$$

whose auxiliary polynomial is

$$x^4 = x^3 + 3x^2 - x - 1 .$$

This agrres with the results in [8].

Now, let $\lambda^2 = x\lambda + 1$. Then

$$f_n(\lambda) = \lambda P_n + Q_n ,$$

when P_n and Q_n are polynomials in λ.

$$\lambda P_{n+2} + Q_{n+2} = \lambda(\lambda P_{n+1} + Q_{n+1}) + \lambda P_n + Q_n$$
$$= x\lambda P_{n+1} + P_{n+1} + \lambda Q_{n+1} + \lambda P_n + Q_n .$$

Thus

$$P_{n+2} = xP_{n+1} + P_n + Q_{n+1} ,$$

$$Q_{n+2} = P_{n+1} + Q_n ,$$

so that, using the same techniques as before, we find

$$P_{n+3} = xP_{n+2} + P_{n+1} + Q_{n+2}$$

$$P_{n+1} = xP_n + P_{n-1} + Q_n$$

$$P_{n+3} - P_{n+1} = xP_{n+2} + P_{n+1} - xP_n - P_{n-1} + (Q_{n+2} - Q_n)$$

yielding

$$P_{n+3} = xP_{n+2} + 3P_{n+1} - xP_n - P_{n-1}$$

which agrees with Eq. (8), particularly result (iii), in [8].

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SOME THEOREMS ON COMPLETENESS

holds true and Theorem 12 is completed.

Corollary. The hypothesis of Theorem 3 is not a necessary condition. From Theorem 7, clearly $F_{n+1}^m \le 2F_n^m$ for $n \ge 3$, $m \ge 4$, and that the sequence 2^{m-1} copies of F_n^m is complete.

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