ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-198 Proposed by E. M. Cohn, National Aeronautics and Space Administration, Washington, D.C.

There is an infinite sequence of square values for triangular numbers,[1]

$$k^2 = m(m+1)/2 .$$

Find simple expressions for k and m in terms of Pell numbers, P_n. ($P_{n+2} = 2P_{n+1} + P_n$, where $P_0 = 0$ and $P_1 = 1$.

H-199 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

A certain country's coinage consists of an infinite number of types of coins: $\cdots, C_{-2}, C_{-1}, C_0, C_1, \cdots$. The value V_n of the coin C_n is related to the others as follows: for <u>all</u> n,

$$V_n = V_{n-3} + V_{n-2} + V_{n-1} .$$

Show that any (finite) pocketful of coins is equal in value to a pocketful containing at most one coin of each type.

H-200 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

Let $M(n)$ be the number of primes (distinct) which divide the binomial coefficient,[2]

$$C_k^n \equiv \binom{n}{k} .$$

[1]A. V. Sylvester, Am. Math. Monthly, 69 (1962), p. 168; quoted in C. W. Trigg, Mathematical Quickies (1967), p. 164.

[2]Divide at least one C_k^n where $0 \leq k \leq n$.

Clearly, for $1 \leq n \leq 15$, we have $M(1) = 0$, $M(2) = M(3) = 1$, $M(4) = M(5) = 2$, $M(6) = M(7) = M(8) = M(9) = 3$, $M(10) = 4$, $M(11) = M(12) = M(14) = 5$, $M(13) = M(15) = 6$, etc. Show that

$$\{M(n)\}_{n=1}^{\infty}$$

has an upper bound and find an asymptotic formula for $M(n)$.

H-201 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Copy 1, 1, 3, 8, $\cdots$, F_{2n-2} $(n \geq 1)$ down in staggered columns as in display C:

$$C \qquad \begin{array}{ccccc} 1 & & & & \\ 1 & 1 & & & \\ 3 & 1 & 1 & & \\ 8 & 3 & 1 & 1 & \\ 21 & 8 & 3 & 1 & 1 \end{array}$$

i) Show that the row sums are F_{2n+1} $(n = 0, 1, 2, \cdots)$

ii) Show that, if the columns are multiplied by 1, 2, 3, $\cdots$ sequentially to the right, then the row sums are F_{2n+2} $(n = 0, 1, 2, \cdots)$.

iii) Show that the rising diagonal sums ($\nearrow$) are F^2_{n+1} $(n = 0, 1, 2, \cdots)$.

H-202 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$\left\{ {k \atop j} \right\} = \frac{F_k F_{k-1} \cdots F_{k-j+1}}{F_1 F_2 \cdots F_j}, \qquad \left\{ {k \atop 0} \right\} = 1 .$$

Show that

$$(\star) \qquad \left\{ \begin{aligned} &\sum_{j=-k}^{k} (-1)^{\frac{1}{2}j(j+1)} \left\{ {2k \atop j+k} \right\} = \prod_{j=1}^{k} L_{2j-1} \\ &\sum_{j=-k}^{k} (-1)^{\frac{1}{2}j(j-1)} \left\{ {2k \atop j+k} \right\} = (-1)^k \prod_{j=1}^{k} L_{2j-1} \end{aligned} \right. ,$$

$$(\star\star) \qquad \left\{ \begin{aligned} &\sum_{j=o}^{2k} (-1)^j \left\{ {2k \atop j} \right\} L_{(j-k)^2} = 2 \cdot 5^{\frac{1}{2}k} F_1 F_3 \cdots F_{2k-1} \qquad (k \text{ even}) \\ &\sum_{j=o}^{2k} (-1)^j \left\{ {2k \atop j} \right\} F_{(j-k)^2} = 2 \cdot 5^{\frac{1}{2}(k-1)} F_1 F_3 \cdots F_{2k-1} \qquad (k \text{ odd}) . \end{aligned} \right.$$

H-203 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

A. Let there be n $(n \geq 1)$ edge-connected squares. How many configurations are there which have each row starting one square to the right of the row above?

B. For the above configurations, how many have each row starting k $(k \geq 0)$ squares to the right of the row above?

H-204 Proposed by Dwarka Nivas, Berhampur, Orissa, India.

Given that 65537 $(= 256^2 + 1)$ is prime, find the remainder when it divides

i) $$\binom{32768}{16384}$$

and

ii) $$16384!$$

SOLUTIONS

NOBODY IS EVEN PERFECT

H-188 Proposed by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Prove that there are no even perfect Fibonacci numbers.

Solution by S. L. Padwa, Applied Mathematics Department, Brookhaven National Laboratory.

As is well known, all even perfect numbers are of the form $2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are prime.

In particular, all even perfect numbers >28 are multiples of 16.

Now the only Fibonacci numbers which are multiples of 16 are also multiples of 9; namely, F_{12k}.

Thus no Fibonacci number which is a multiple of 16 is of the required form for perfect numbers since they are all multiples of 9, while even perfect numbers cannot have an odd composite factor.

Also solved by the Proposer.

SOME SUMS

H-191 Proposed by David Zeitlin, Minneapolis, Minnesota

Prove the following identities:

(a) $$\sum_{k=0}^{2n} \binom{2n}{k}^3 L_{2k} = L_{2n} \sum_{k=0}^{n} \frac{(2n+k)!}{(k!)^3(2n-2k)!} 5^{n-k}$$

(b) $$\sum_{k=0}^{2n+1}\binom{2n+1}{k}^3 L_{2k} = F_{2n+1}\sum_{k=0}^{n}\frac{(2n+1+k)!}{(k!)^3(2n+1-2k)!}\,5^{n+1-k}$$

(c) $$\sum_{k=0}^{2n}\binom{2n}{k}^3 F_{2k} = F_{2n}\sum_{k=0}^{n}\frac{(2n+k)!}{(k!)^3(2n-2k)!}\,5^{n-k}$$

(d) $$\sum_{k=0}^{2n+1}\binom{2n+1}{k}^3 F_{2k} = L_{2n+1}\sum_{k=0}^{n}\frac{(2n+1+k)}{(k!)^3(2n+1-2k)!}\,5^{n-k},$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

Solution by the Proposer.

From the solution to H-180, we recall that

(1) $$\sum_{k=0}^{p}\binom{p}{k}^3 x^k = \sum_{2k\le p}\frac{(p+k)!}{(k!)^3(p-2k)!}\,x^k(x+1)^{p-2k}.$$

Since $\alpha^2+1 = \alpha^2-\alpha\beta = \alpha(\alpha-\beta)$ and $\beta^2+1 = \beta(\beta-\alpha)$

$$\left(\alpha = \frac{1+\sqrt{5}}{2},\quad \beta = \frac{1-\sqrt{5}}{2}\right),$$

we obtain from (1) for $x = \alpha^2$ and $x = \beta^2$,

(2) $$\sum_{k=0}^{p}\binom{p}{k}^3 \alpha^{2k} = \sum_{2k\le p}\frac{(p+k)!}{(k!)^3(p-2k)!}\,\alpha^{2k}[\alpha(\alpha-\beta)]^{p-2k}$$

and

(3) $$\sum_{k=0}^{p}\binom{p}{k}^3 \beta^{2k} = \sum_{2k\le p}\frac{(p+k)!}{(k!)^3(p-2k)!}\,\beta^{2k}[-\beta(\alpha-\beta)]^{p-2k},$$

respectively. So, by addition of (2), (3), we get

$$\sum_{k=0}^{p}\binom{p}{k}^3 L_{2k} = [\alpha^p + (-1)^p\beta^p]\sum_{2k\le p}\frac{(p+k)!}{(k!)^3(p-2k)!}\,5^{\frac{p-2k}{2}},$$

since $(\alpha - \beta)^2 = 5$ and $\alpha^{2k} + \beta^{2k} = L_{2k}$. For $p = 2n$, we obtain (a); for $p = 2n + 1$, we obtain (b). By subtraction of (3) from (2), we get

$$\sum_{k=0}^{p} \binom{p}{k}^3 (\alpha^{2k} - \beta^{2k}) = \alpha^p - (-1)^p \beta^p \sum_{2k \le p} \frac{(p+k)!}{(k!)^3 (p-2k)!} 5^{\frac{p-2k}{2}},$$

since $(\alpha^{2k} - \beta^{2k})/(\alpha - \beta) = F_{2k}$.

For $p = 2n$, we obtain (c); for $p = 2n + 1$, we obtain (d).

SECOND DEGREE FOR DIOPHANTUS

H-194 Proposed by H. V. Krishna, Manipal Engineering College, Manipal, India.

Solve the Diophantine equations:

(i) $$x^2 + y^2 \pm 5 = 3xy$$

(ii) $$x^2 + y^2 \pm e = 3xy,$$

where

$$e = p^2 - pq - q^2;$$

p, q positive integers.

Solution by the Proposer.

Rewrite (ii) as

(1) $$(x + y)^2 - 5xy = \pm e .$$

Let $H_0 = q$, $H_1 = p$, and $H_{n+2} = H_{n+1} + H_n$, $n \ge 0$ be the generalized Fibonacci sequence. Then we have the following identities viz.

(2) $$(H_{2r-1} + H_{2r+1})^2 - 5H_{2r-1}H_{2r+1} = (-1)^{2r-1}e$$

and

(3) $$(H_{2r} + H_{2r+2})^2 - 5H_{2r}H_{2r+2} = (-1)^{2r}e ,$$

from which the solution of (ii) easily follows. (i) is a particular case, where $e = -5$.

EDITORIAL NOTES

Correction to H-185.

Show that

$$(1 - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \binom{2k}{k} (1 - x)^{n-k} {}_2F_1[-k, n + k + 1; k + 1; x] ,$$

where ${}_2F_1[a, b; c; x]$ denotes the hypergeometric function.

Comment on H-193.

The proposer has pointed out that the stated condition does hold for the following examples.

Examples: $5 + 1 + 1 = 7 = 2^3 - 1, \quad 5^3 + 1^3 + 1^3 = 127 = 2^7 - 1,$

$$19 + 11 + 1 = 31 = 2^5 - 1, \quad 19^3 + 11^3 + 1^3 = 8191 = 2^{13} - 1,$$

$$79 + 29 + 19 = 127 = 2^7 - 1, \quad 79^3 + 29^3 + 19^3 = 524287 = 2^{19} - 1.$$

The validity of the statement would be a pleasant surprise.

Late Acknowledgements

H-183 P. Lindstrom, D. Klarner, S. Smith, D. Priest, and L. Carlitz.

Notice: The editor would be happy to override the "two months after publication" clause for solutions of problems prior to H-180, for which no solutions have been published. The next issue will contain a complete list of unresolved problems. Please send your solutions!

[Continued from page 590.]

$$\begin{aligned} 2\alpha &= \theta - \psi, & 2\beta &= \theta + \psi, \\ x &= 2\cos\theta, & y &= 2\cos\psi, \\ z &= xy + 2, & a &= \frac{1}{2}(x + y). \end{aligned}$$

We shall consider the asymmetric five diagonal determinant on another occasion.

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