ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

NOTATION: $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$; $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-244 Proposed by J. L. Hunsucker, University of Georgia, Athens, Georgia.

Let Q be the 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the sum of a finite number of matrices chosen from the sequence $Q, Q^2, Q^3, \cdots$. Prove that $b = c$ and $a = b + d$.

B-245 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that each term F_n with $n > 0$ in the sequence $F_0, F_1, F_2, \cdots$ is expressible as $x^2 + y^2$ or $x^2 - y^2$ with x and y terms of the sequence with distinct subscripts.

B-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that at least one of the following sums is irrational:

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}.$$

B-247 Proposed by Larry Lang, Student, San Jose State University, San Jose, California.

Given that m and n are integers with $0 < n < m$ and $F_n \mid L_m$, prove that n is 1, 2, 3, or 4.

B-248 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let k be a positive integer and let $h = 5^k$. Prove that $h \mid F_h$.

B-249 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let k be a positive integer and let $g = 2 \cdot 3^k$. Prove that $g \mid L_g$.

SOLVERS INADVERTENTLY OMITTED FROM PREVIOUS ISSUES

J. L. Brown, Jr: B-219

Herta T. Freitag: B-202, B-203, B-206, B-207.

D. V. Jaiswal: B-214, B-215, B-216, B-217, B-218, B-219.

Graham Lord: B-202, B-203, B-204, B-205

SOLUTIONS

TWIN PRIMES SLIGHTLY DISGUISED

B-220 Proposed by Guy A. R. Guillotte, Montreal, P. Q., Canada.

Let p_m be the m^{th} prime. Prove that p_m and p_{m+1} are twin primes (i.e., $p_{m+1} = p_m + 2$) if and only if

$$\sum_{n=1}^{m} (p_{n+1} - p_n) = p_m .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

The sum telescopes to $p_{m+1} - p_1 = p_{m+1} - 2$.

Also solved by Wray G. Brady, Paul S. Bruckman, Warren Cheves, R. Garfield, Herta T. Freitag, Peter A. Lindstrom, Graham Lord, John W. Milsom, Richard W. Sielaff, and the Proposer.

SIMPLE SUBSTITUTION IN A CONVERGENT SERIES

B-221 Proposed by R. Garfield, College of Insurance, New York, New York.

Prove that

$$\sum_{n=1}^{\infty} (1/F_nL_n) = \sum_{n=1}^{\infty} (1/F_{2n}) .$$

Solution by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

It follows from the identity $F_{2n} = F_n L_n$ that the series are identical. Since

$$\lim_{n\to\infty} (F_{2n}/r^{2n}) = 1$$

where

$$r = (1 + \sqrt{5})/2 > 1 ,$$

the series converge.

Also solved by Paul S. Bruckman, Herta T. Freitag, Peter A. Lindstrom, Graham Lord, C. B. A. Peck, Richard W. Sielaff, Gregory Wulczyn, and the Proposer.

A NONHOMOGENEOUS RECURSION

B-222 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Find a formula for K_n where $K_1 = 1$ and

$$K_{n+1} = (K_1 + K_2 + \cdots + K_n) + F_{2n+1} .$$

Solution by C. B. A. Peck, State College, Pennsylvania.

Reducing the subscript $n+1$ to n in the given recursion, we have $K_n = (k_1 + K_2 + \cdots + K_{n-1}) + F_{2n-1}$. Subtracting from corresponding sides of the original gives us

$$K_{n+1} - K_n = K_n + F_{2n+1} - F_{2n-1} = K_n + F_{2n}$$

or

$$K_{n+1} = 2K_n + F_{2n} .$$

Then $K_1 = 1$, $K_2 = 2\cdot 1 + 1$, $K_3 = 2^2\cdot 1 + 2\cdot 1 + 3$, and generally

$$K_n = 2^{n-1} + (2^{n-2}F_2 + 2^{n-3}F_4 + \cdots + 2F_{2n-4} + F_{2n-2}) .$$

Using a result of Herta T. Freitag (Fibonacci Quarterly, Vol. 8, No. 5, p. 344), we have

$$K_n = F_{2n+1} - 2^{n-1} .$$

Also solved by Paul S. Bruckman, L. Carlitz, Herta T. Freitag, Graham Lord, David Zeitlin, Gregory Wulczyn, and the Proposer.

FORMIDABLE ARITHMETIC

B-223 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona.

Find a solution of

$$x^y + (x + 3)^y - (x + 4)^y = u^v + (u + 3)^v - (u + 4)^v$$

in the form

$$x = F_m, \quad y = F_n, \quad u = L_r, \quad \text{and} \quad v = L_s .$$

Solution by the Proposer.

A solution is

$$x = 13 = F_7, \quad y = 5 = F_5, \quad u = 7 = L_4, \quad \text{and} \quad v = 3 = L_2 .$$

QUADRATIC NONRESIDUES

B-224 Proposed by Lawrence Somer, Champaign, Illinois

Let m be a fixed positive integer. Prove that no term in the sequence $F_1, F_3, F_5, F_7, \cdots$ is divisible by $4m - 1$.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Since

$$5F_{2n+1}^2 - 4 = L_{2n+1}^2 ,$$

it would follow from

$$F_{2n+1} \equiv 0 \pmod{4m - 1}$$

that

$$L_{2n+1}^2 \equiv -4 \pmod{4m - 1} .$$

This implies the solvability of the congruence

$$x^2 \equiv -1 \pmod{4m - 1}$$

which is impossible.

Also solved by Paul S. Bruckman, Graham Lord, and the Proposer.

STILL UNCHARACTERIZED SEQUENCES

B-225 Proposed by John Ivie, Berkeley, California.

Let $a_0, \cdots, a_{j-1}$ be constants and let $\{f_n\}$ be a sequence of integers satisfying

$$f_{n+j} = a_{j-1}F_{n+j-1} + a_{j-2}f_{n+j-2} + \cdots + a_o f_n ; \qquad n = 0, 1, 2, \cdots .$$

Find a necessary and sufficient condition for $\{f_n\}$ to have the property that every integer m is an exact divisor of some f_k.

EDITORIAL NOTE: A necessary and sufficient condition that m divides some f_k is that m divides some f_k for $1 \leq k \leq m^2 + 1$, for every m.