# CONVOLUTION TRIANGLES

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If $G(x)$ is the generating function for a sequence, then $(G(x))^{k+1}$ is the column generator for the $k^{th}$ column of the CONVOLUTION TRIANGLE. The original sequence is the zeroth column. We study here the convolution triangle of a class of generalized Fibonacci sequences which are obtained as rising diagonal sums of generalized Pascal triangles induced by the expansions $(1 + x + x^2 + \cdots + x^{r-1})^n$, $n = 0, 1, 2, 3, \cdots$. There are several ways to generate the convolution triangle array for a given generalized Fibonacci sequence. We shall illustrate these with the Fibonacci sequence.

## 1. GENERATION OF ARRAYS

In [1], it is shown that a rule of formation for the Fibonacci convolution triangle is as follows: to get the $n^{th}$ element in the $k^{th}$ column, add the two elements above it in the same column and the one immediately to the left in the preceding column. One notes in passing that this is equivalent to the following: Start row zero with a row of ones extending to the right. To get an element $A$ in this array, add the two elements directly above $A$ and all those elements in the same two rows and to the left of these.

(1.1)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | $\cdots$ | 1 | 1 | $\cdots$ |
| 1 | 2 | 3 | 4 | 5 | $\cdots$ | m | m + 1 | $\cdots$ |
| 2 | 5 | 9 | 14 | 20 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| 3 | 10 | 22 | 40 | 65 | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| 5 | 20 | 51 | 105 | 190 | $\cdots$ | x | $\cdots$ | $\cdots$ |
| 8 | 38 | 111 | 256 | 511 | $\cdots$ | y | | |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | A | | |

It is easy to prove by mathematical induction that this generates the Fibonacci convolution array.

One might observe that the rising diagonal sums in the array above are the Pell sequence $P_1 = 1$, $P_2 = 2$, $P_{n+2} = 2P_{n+1} + P_n$. The rising diagonal sums formed by going up 2 and over 1 are $1, 1, 3, 5, 11, 21, \cdots, u_{n+2} = u_{n+1} + 2u_n$. Also, the determinant of the square arrays of order 1, 2, 3, 4, 5, readily found in the left-hand corner of the array, is in each case equal to one.

By comparison, if Pascal's triangle is written in rectangular form

(1.2)

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | ⋯ |
| 1 | 2 | 3 | 4 | 5 | ⋯ |
| 1 | 3 | 6 | 10 | 15 | ⋯ |
| 1 | 4 | 10 | 20 | 35 | ⋯ |
| 1 | 5 | 15 | 35 | 70 | ⋯ |
| ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ |

the rule of formation to obtain an element A is to add the one element above A and all elements in that same one row, or, to add the one element above A and the element in the preceding column to the left of A. The rising diagonal sums are the powers of two, and the sums of diagonals formed by going up 2 and over 1 are the Fibonacci numbers 1, 1, 2, 3, 5, 8, $\cdots$, $F_{n+2} = F_{n+1} + F_n$.

When we speak of rising diagonal sums in generalized Pascal's triangles we are thinking of diagonals formed by going up 2 and over 1 in rectangular arrays similar to (1.2) or going up 1 and right 1 in a left-justified array such as

(1.3)

| | | | | | |
|---|---|---|---|---|---|
| 1 | | | | | |
| 1 | 1 | | | | |
| 1 | 2 | 1 | | | |
| 1 | 3 | 3 | 1 | | |
| 1 | 4 | 6 | 4 | 1 | |
| ⋯ | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ |

The coefficients of the Fibonacci polynomials

$$f_0(x) = 0, \qquad f_1(x) = 1, \qquad f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$$

lie along the rising diagonals of Pascal's triangle (1.3) and $f_n(1) = F_n$. It is well known that the generating function for the Fibonacci polynomials is

$$\frac{\lambda}{1 - x\lambda - \lambda^2} = \sum_{n=0}^{\infty} f_n(x)\lambda^n .$$

$$\frac{d^m}{dx^m}\left(\frac{\lambda}{1 - x\lambda - \lambda^2}\right) = \frac{\lambda^{m+1}\, m!}{(1 - x\lambda - \lambda^2)^{m+1}} = \sum_{n=0}^{\infty} f_n^{(m)}(x)\lambda^n .$$

Since $f_n(x)$ is of degree $n - 1$, setting $x = 1$ yields

$$\frac{\lambda^{m+1}}{(1-\lambda-\lambda^2)^{m+1}} = \sum_{n=0}^{\infty} \frac{f_n^{(m)}(1)}{m!}\lambda^n$$

so that since

$$\left(\frac{\lambda}{1-\lambda-\lambda^2}\right)^{m+1} = \sum_{n=0}^{\infty} F_n^{(m)}\lambda^n$$

also generates the $m^{th}$ convolution sequence, equating coefficients shows that

$$\frac{f_n^{(m)}(1)}{m!} = F_n^{(m)} ,$$

where $F_n^{(m)}$ is the $n^{th}$ member of the $m^{th}$ Fibonacci convolution sequence. Thus the Fibonacci polynomials $f_n(x)$ evaluated at $x = 1$ by the Taylor's series expansion have as coefficients elements that lie along diagonals of the Fibonacci convolution triangle (1.1), which are the rows of (1.4):

(1.4)

| | $f_n(1)$ | $f'_n(1)/1!$ | $f''_n(1)/2!$ | $f'''_n(1)/2!$ |
|---|---|---|---|---|
| $f_1(x)$ | 1 | 0 | 0 | 0 |
| $f_2(x)$ | 1 | 1 | 0 | 0 |
| $f_3(x)$ | 2 | 2 | 1 | 0 |
| $f_4(x)$ | 3 | 5 | 3 | 1 |
| $f_5(x)$ | 5 | 10 | 9 | 4 |
| $f_6(x)$ | 8 | 20 | 22 | ... |
| ... | ... | ... | ... | ... |

## 2. THE JACOBSTHAL POLYNOMIALS

Consider the polynomials $J_1(x) = 1$, $J_2(x) = 1$, and $J_{n+2}(x) = J_{n+1}(x) + xJ_n(x)$. We see, of course, that $J_n(1) = F_n$. The coefficients of the Jacobsthal polynomials also lie on the rising diagonals of Pascal's triangle (1.3) but their order is the reverse of that for the Fibonacci polynomials. The generating function for the Jacobsthal polynomials is

$$\frac{\lambda}{1-\lambda-x\lambda^2} = \sum_{n=0}^{\infty} J_n(x)\lambda^n$$

from which

$$\frac{d^m}{dx^m}\left(\frac{\lambda}{1-\lambda-x\lambda^2}\right) = \frac{\lambda^{2m+1}\, m!}{(1-\lambda-x\lambda^2)^{m+1}} = \sum_{n=0}^{\infty} J_n^{(m)}(x)\lambda^n$$

so that

$$\frac{\lambda^{m+1}}{(1-\lambda-\lambda^2)^{m+1}} = \sum_{n=0}^{\infty} \frac{J_n^{(m)}(1)}{m!}\lambda^{n-m} .$$

Thus

$$\frac{J_n^{(m)}(1)}{m!} = F_n^{(m)} ,$$

also.

The Jacobsthal polynomial sequence has two polynomials of each degree. The array obtained by listing the polynomials and their derivatives at $x = 1$ appropriately divided by $m!$ also yields the Fibonacci convolution array.

There is another nameless set of polynomials that has interesting related properties, $Q_1(x) = 1$, $Q_2(x) = x$, and $Q_{n+2}(x) = x(Q_{n+1}(x) + Q_n(x))$. These polynomials also have their coefficients along the rising diagonals of Pascal's triangle (1.3). Clearly, thus $Q_n(1) = F_n$. The generating function for the $Q_n(x)$ is

$$\frac{\lambda}{1 - x(\lambda + \lambda^2)} = \sum_{n=0}^{\infty} Q_n(x)\lambda^n ;$$

$$\frac{(1+\lambda)^m \lambda^{m+1}}{(1-\lambda-\lambda^2)^{m+1}} = \sum_{n=0}^{\infty} \frac{Q_n^{(m)}(1)}{m!}\lambda^n .$$

We will leave the reader to do the analysis of the array obtained from $Q_n^{(m)}(1)/m!$.

## 3. ROW GENERATING FUNCTIONS

For the Fibonacci convolution array the column generators are

$$\left(\frac{\lambda}{1-\lambda-\lambda^2}\right)^{k+1} \qquad k = 0, 1, 2, 3, \cdots .$$

Here we are interested in the row generators when the array is written in the form (1.1) which starts with a row of ones. Since an element A of that array is secured by adding the two elements in the column above A and the element in the preceding column directly to the left of A, we now wish to secure the row generating functions based on this same generating scheme. Let $R_n(x)$ be the row generating function; then the recurrence scheme dictates that

$$R_{n+2}(x) = xR_{n+2}(x) + R_{n+1}(x) + R_n(x) .$$

We note that

$$R_0(x) = \frac{1}{1-x} \qquad \text{and} \qquad R_1(x) = \frac{1}{(1-x)^2} ,$$

thus since

$$R_{n+2}(x) = \frac{R_{n+1}(x) + R_n(x)}{1 - x}$$

it follows that the general form of

$$R_n(x) = \frac{N_n(x)}{(1 - x)^{n+1}} ,$$

where the numerator polynomials $N_n(x)$ obey the recurrence

$$N_{n+2}(x) = N_{n+1}(x) + (1 - x)N_n(x)$$

with $N_0(x) = 1$ and $N_1(x) = 1$. Surely now we recall the Jacobsthal polynomials discussed earlier and we observe that

$$J_{n+1}(1 - x) = N_n(x) .$$

Expand the polynomials $J_n(x)$ in a Taylor's series about $x = 1$ to yield

$$J_n(x) = J_n(1) + \frac{J_n'(1)}{1!}(x - 1) + \frac{J_n''(1)}{2!}(x - 1)^2 + \cdots$$

$$J_n(1 - x) = J_n(1) - \frac{J_n'(1)}{1!} x + \frac{J_n''(1)}{2!} x^2 - \frac{J_n'''(1)}{3!} x^3 + \cdots$$

Thus we conclude that

$$N_n(x) = J_{n+1}(1 - x)$$

are polynomials whose coefficients lie along the rising diagonals of the Fibonacci convolution triangle (1.4) whose modified column generators are

$$-\left(\frac{-\lambda}{1 - \lambda - \lambda^2}\right)^{m+1} \qquad m = 0, 1, 2, \cdots .$$

## 4. THE GENERALIZATION OF THE FIBONACCI CONVOLUTION ARRAYS

Consider the arrays whose column generators are

$$\left(\frac{\lambda}{1 - \lambda - \lambda^2 - \cdots - \lambda^r}\right)^{m+1} \qquad m = 0, 1, 2, 3, \cdots .$$

These are convolution arrays for those rising diagonal sum sequences in the generalized Pascal's triangle induced by

$$(1 + x + x^2 + \cdots + x^{r-1})^n, \qquad n = 0, 1, 2, \cdots$$

and written in a left-justified manner as is (1.3). Such sequences are called the generalized Fibonacci sequences.

We will illustrate the generalization using the Tribonacci sequence

$$T_1 = T_2 = 1, \quad T_3 = 2, \quad \cdots, \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad n = 1, 2, 3, \cdots.$$

The Tribonacci convolution triangle written in rectangular form is

(4.1)

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | ⋯ |
| 1 | 2 | 3 | 4 | 5 | ⋯ |
| 2 | 5 | 9 | 14 | 20 | ⋯ |
| 4 | 12 | 25 | 44 | 70 | ⋯ |
| 7 | 26 | 63 | 125 | 220 | ⋯ |
| 13 | 56 | ⋯ | ⋯ | ⋯ | ⋯ |
| 24 | ⋯ | ⋯ | ⋯ | ⋯ | ⋯ |
| ⋯ | | | | | |

The rising diagonal sums of the Tribonacci array (4.1) obey the recurrence

$$U_{n+3} = 2U_{n+2} + U_{n+1} + U_n ,$$

where

$$U_1 = 1, \quad U_2 = 2, \quad U_3 = 5, \quad U_4 = 13, \quad \cdots$$

These could be called the generalized Pell sequence corresponding to the Trinomial triangle, The diagonals formed by going up 2 and right 1 in the Pascal case were Fibonacci numbers as sums; in the Fibonacci case, going up 3 and right 1 gave Tribonacci numbers; here the diagonals formed by going up 4 and right 1 in (4.1) give Quadranacci numbers, 1, 1, 2, 4, 8, 15, 29, ⋯, where $u_{n+4} = u_{n+3} + u_{n+2} + u_{n+1} + u_n$.

The corresponding Jacobsthal polynomials for the trinomial triangle are given by

$$J^{\star}_{n+3}(x) = J^{\star}_{n+2}(x) + xJ^{\star}_{n+1}(x) + x^2J^{\star}_n(x)$$

with

$$J^{\star}_1(x), \qquad J^{\star}_2(x) = 1, \qquad J^{\star}_3(x) = 1 + x$$

The generating function for the generalized Jacobsthal polynomials is

$$\frac{\lambda}{1 - \lambda - x\lambda - x^2\lambda^2} = \sum_{n=0}^{\infty} J_n^*(x)\lambda^n$$

It is not hard to prove that the row generating functions for the Tribonacci convolution array (4.1) are generally

$$R_n^*(x) = \frac{N_n^*(x)}{(1 - x)^{n+1}} ,$$

where

$$R_0^*(x) = \frac{1}{1 - x}, \quad R_1^*(x) = \frac{1}{(1 - x)^2}, \quad R_2^*(x) = \frac{2 - x}{(1 - x)^3}$$

Thus one asserts the polynomials $N_n^*(x)$ obey

$$N_{n+3}^*(x) = N_{n+2}^*(x) + (1 - x)N_{n+1}^*(x) + (1 - x)^2 N_n^*(x) ,$$

where

$$N_0^*(x) = 1, \quad N_1^*(x) = 1, \quad \text{and} \quad N_2^*(x) = 2 - x .$$

Further,

$$J_{n+1}^*(1 - x) = N_n^*(x)$$

and

$$N_n^*(x) = J_{n+1}^*(1) - \frac{J_{n+1}^{*\prime}(1)}{1!} x + \frac{J_{n+1}^{*\prime\prime}(1)}{2!} x^2 - \cdots$$

the same as before (this has alternating signs).

There are several polynomial sequences yielding the Tribonacci convolution array when one generates $P_n^{(m)}(1)/m!$, where $P_n(x)$ are the generalized Fibonacci polynomials, or Tribonacci polynomials,

$$P_{n+3}(x) = x^2 P_{n+2}(x) + xP_{n+1}(x) + P_n(x) ,$$

where

$$P_1(x) = P_2(x) = 1 \quad \text{and} \quad P_3(x) = 1 + x .$$

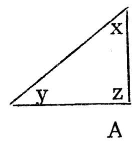
One such example:

$$\frac{\lambda}{1 - x\lambda - \lambda^2 - \lambda^3} = \sum_{n=0}^{\infty} P_n(x)\lambda^n$$

$$\frac{\lambda^{m+1} m!}{(1 - x\lambda - \lambda^2 - \lambda^3)^{m+1}} = \sum_{n=0}^{\infty} P_n^{(m)}(x)\lambda^n$$

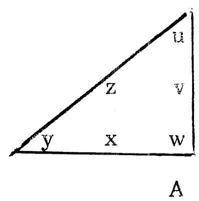
$$\left(\frac{\lambda}{1 - \lambda - \lambda^2 - \lambda^3}\right)^{m+1} = \sum_{n=0}^{\infty} T_n^{(m)} \lambda^n = \sum_{n=0}^{\infty} \frac{P_n^{(m)}(1)}{m!} \lambda^n$$

Recall from Section 1 that the convolution triangle (1.4) for the Fibonacci numbers is generated by adding x, y, and z to get element A as in the diagram below:




$$A = x + y + z \quad .$$

Recall this is also the array generated by the numerator polynomials for the row generators of the Fibonacci convolution array. For the Tribonacci convolution array row generators can also be self-generated if element $A = u + v + w + x + y + 2z$ where the elements u, v, w, x, y, and z are found by the diagram




$$A = u + v + w + x + y + 2z$$

beginning the array with a one. Here the coefficients of the numerator polynomials of the row generators of the Tribonacci array lie along the rising diagonals of the triangle array below, which has the Tribonacci numbers in its left column. Of course, one normally asks what are the column generators of this triangle, too.

(4.2)

| | | | | | | |
|---|---|---|---|---|---|---|
| | 1 | | | | | |
| | 1 | 1 | 1 | | | |
| | 2 | 4 | 3 | 2 | 1 | |
| | 4 | 9 | 12 | 11 | 6 | ... |
| $R_4^*$ ——— | 7 | 22 | 37 | 40 | ... | ... |
| | 13 | 50 | ... | ... | ... | ... |
| | 24 | ... | ... | ... | ... | ... |
| | ... | ... | ... | ... | ... | ... |

Here we shall illustrate several row generators:

$$R_0^*(x) = \frac{1}{1 - x}$$

$$R_1^*(x) = \frac{1}{(1 - x)^2}$$

$$R_2^*(x) = \frac{2 - x}{(1 - x)^3}$$

$$R_3^*(x) = \frac{4 - 4x + x^2}{(1 - x)^4}$$

$$R_4^*(x) = \frac{7 - 9x + 3x^2}{(1 - x)^5}$$

The column generators of (4.2) are

$$G_{n+2}(x) = \left(\frac{x}{1 - x - x^2 - x^3}\right)\left((2x + 1)G_{n+1}(x) + G_n(x)\right),$$

$$G_0(x) = \frac{1}{1 - x - x^2 - x^3}, \qquad G_1(x) = \frac{x(1 + 2x)}{(1 - x - x^2 - x^3)^2} .$$

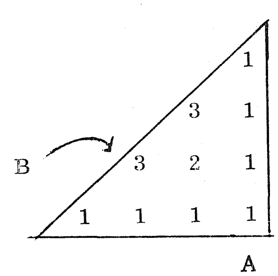
## 5. THE FULL GENERALIZATION

The generalized Jacobsthal polynomials are $J_1(x) = 1$, $J_2(x) = 1$, $J_3(x) = 1 + x$, $J_4(x) = (1 + x)^2$, $\cdots$, $J_{m+2}(x) = (1 + x)^m$, and

$$J_{n+m+1}(x) = J_{n+m}(x) + xJ_{n+m-1}(x) + x^2J_{n+m-2}(x) + \cdots + x^mJ_n(x)$$

$$= \sum_{j=0}^{m} J_{n+m-j}(x)x^{j-1} .$$

The numerator polynomial triangle is constructed by taking the appropriate size triangle B above A where the multipliers for the elements in B are the elements in the first k rows of Pascal's triangle as illustrated for k = 4 below:




The left edge of this triangular array is the Quadranacci sequence:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 1 | 1 | 1 | 1 | | | |
| 2 | 4 | 6 | 4 | 3 | 2 | 1 |
| 4 | 12 | 18 | 22 | 22 | 18 | ... |
| 8 | 28 | 58 | 88 | 106 | 100 | ... |
| 15 | 67 | ... | ... | ... | ... | ... |
| 29 | 154 | ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... | ... | ... |

In each case, the generalized Pascal's array can be generated by adding all the elements in the rectangle with k rows above and to the left of element A (not including elements in the same column as A) to get A. If the rectangle has k rows, then we get the array induced by the expansions $(1 + x + x^2 + \cdots + x^{k-1})^n$, $n = 0, 1, 2, \cdots$. In these rectangular arrays using k rows in formation, if sums are found of elements lying on diagonals formed by going up $(k + 1)$ and right one, the sequence formed obeys the recurrence

$$u_{n+k+1} = u_{n+k} + u_{n+k-1} + \cdots + u_n .$$

where $u_1 = u_2 = 1$, $u_n = 2^{n-2}$ for $2 \le n \le k + 1$, generalized Fibonacci sequences, while the rising diagonals yield sums which are generalized Pell sequences obeying the recurrence

$$p_{n+k} = 2p_{n+k-1} + (p_{n+k-2} + p_{n+k-3} + \cdots + p_n)$$

and with the first three members of the sequence the ordinary Pell numbers 1, 2, 5, and the first k members of the sequence the same as the first k members of the sequence found from the rectangular array using $(k - 1)$ rows in its formation.

The convolution triangle for such generalized Fibonacci sequences can be generated by adding all the elements in the rectangle with k rows, including the column above an element A and extending to the extreme left of the array.

In any of these generalized Pascal's arrays or convolution arrays of generalized Fibonacci sequences written in rectangular form, the determinant of any square array found in the upper left-hand corner is always equal to one. The proofs and extensions will appear in later papers [2], [3].

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