CONVERGENCE OF THE COEFFICIENTS IN THE k^{th} POWER OF A POWER SERIES

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1. CONVOLUTED SUM FORMULAS

In this paper we investigate generalized convoluted numbers and sums by using recurring power series

$$(1) \qquad \left(1 + \sum_{v=1}^{m} a_v x^v\right)^{-k} = \sum_{n=0}^{\infty} u(n,k,m)\, x^n ,$$

where the coefficients a_v and $u(n,k,m)$ are rational integers $k = 1, 2, 3, \cdots$, $u(0,k,m) = 1$ and $m = 1, 2, 3, \cdots$.

By elementary means, it is easy to prove, if

$$(2) \qquad (1 - y)^{-k} = \sum_{v=0}^{\infty} b_v^{(k)} y^v$$

then

$$\binom{n + k - 1}{k - 1} = b_n^{(k)} ,$$

where

$$b_0^{(k)} = 1, \quad k = 1, 2, 3, \cdots, \quad n = 0, 1, 2, \cdots,$$

and

$$\binom{n + k - 1}{k - 1} = (n + k - 1)!/n!\,(k - 1)! \ .$$

Elsewhere [1], it has been shown that the following convoluted sum formulas hold:

$$(3) \qquad u(n,k,2) = \sum_{j=0} \binom{n + k - 1 - j}{k - 1} \binom{n - j}{j} a_1^{n-2j} a_2^j$$

$$(n = 0, 1, 2, \cdots, \ k = 1, 2, 3, \cdots) ;$$

and

$$(4) \qquad u(n,k,3) = \sum_{r=0} \sum_{j=0}^{r} \left[\binom{k + n - 2r - 2}{k - 1} \binom{n - 2r - 1}{2r + 1 - j} \binom{2r + 1 - j}{j} a_1^S a_2^T a_3^j \right.$$

$$\left. + \binom{k + n - 2r - 1}{k - 1} \binom{n - 2r}{2r - j} \binom{2r - j}{j} a_1^{S+2} a_2^{T-1} a_3^j \right]$$

where $S = n - 4r - 2 + j$, $T = 2r + 1 - 2j$, $n = 0, 1, 2, \cdots$, and $k = 1, 2, 3, \cdots$.

The $u(n,k,2)$ in (3) are called "generalized Fibonacci numbers," the $u(n,k,3)$ in (4) are called "generalized Tribonacci numbers," we shall term the $u(n,k,4)$ as the "generalized Quatronacci numbers," and the general expression $u(n,k,m)$ in (1 for $m = 5, 6, \cdots$) we shall refer to as the "generalized Multinacci numbers."

Now in (2) we let

$$y = \sum_{w=1}^{m} a_w x^w \qquad (m = 2, 3, \cdots)$$

and put

$$(1 - y)^{-k} = \sum_{n=0}^{\infty} u(n,k,m)x^n = \sum_{v=0}^{\infty} b_v^{(k)} y^v , \tag{5}$$

and by comparing the coefficients in (5), it is easy to prove with induction, that

$$\sum_{r_1=0} \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \cdots \sum_{r_{m-1}=0}^{r_{m-2}} \phi(n,m)F(n,m)\, b_{n-r_1}^{(k)} = u(n,k,m) , \tag{6}$$

where

$$\phi(n,m) = \binom{n - r_1}{r_1 - r_2}\binom{r_1 - r_2}{r_2 - r_3}\cdots\binom{r_{m-3} - r_{m-2}}{r_{m-2} - r_{m-1}}\binom{r_{m-2} - r_{m-1}}{r_{m-1}} ,$$

$$F(n,m) = a_1^{n-2r_1+r_2}\, a_2^{r_1-2r_2+r_3} \cdots a_{m-2}^{r_{m-3}-2r_{m-2}+r_{m-1}}\, a_{m-1}^{r_{m-2}-2r_{m-1}}\, a_m^{r_{m-1}} ,$$

$$b_{n-r_1}^{(k)} = \binom{n + k - r_1 - 1}{1 - 1} ,$$

and $n = 0, 1, 2, \cdots$, $m = 2, 3, 4, \cdots$.

Of course the convoluted sum formula of the generalized Quatronacci number $u(n,k,4)$ is immediate as a special case of (6, with $m = 4$).

2. A GENERAL METHOD TO FIND FORMULAS FOR THE $u(n,k,m)$ AS A FUNCTION OF $u(j,1,m)$ $(n,j = 0, 1, 2, \cdots)$

In [1], it has been shown that the following formulas for the generalized Fibonacci numbers hold:

$$(a_1^2 + 4a_2)ku(n - 1, k + 1, 2) = a_1 nu(n, k, 2) + a_2(4k + 2n - 2)u(n - 1, k, 2) , \tag{7}$$

where $u(0,k,2) = 1$, $u(1,k,2) = a_1 k$, and $n,k = 1, 2, 3, \cdots$.

Now, using the results in (7) we are able to write the following: where

$A = a_1^2 + 4a_2$, $B(k,n) = 4k + 2n - 2$, $u(0,k,2) = 1$, $u(1,k,2) = a_1k$, $n,k = 1, 2, 3, \cdots$, and

$$u(n,1,2) = u(n - 1, 1, 2)a_1 + u(n - 2, 1, 2)a_2 ,$$

(where a_1 and a_2 are rational integers) we have

(8) $$u(n - 1, 2, 2)A = u(n,1,2)na_1 + u(n - 1, 1, 2)B(1,n)a_2 ,$$

(8.1) $$u(n - 1, 3, 2)A^2 2! = (a_1a_2nB(1, n + 1) + a_1a_2nB(2,n) + a_1^3n(n + 1))u(n,1,2) + (a_2^2B(1,n)B(2,n) + a_1^2a_2n(n + 1))u(n - 1,1,2),$$

and

(8.2) $$u(n - 1, 4, 2)A^3 3! = M + N ,$$

where

$$M = \begin{bmatrix} a_1a_2^2nB(1, n + 1)B(3,n) + a_1a_2^2nB(2,n)B(3,n) \\ + a_1^3a_2n(n + 1)B(3,n) + a_1a_2^2nB(1, n + 1)B(2, n + 1) \\ + a_1^3a_2n(n + 1)(n + 2) + a_1^3a_2n(n + 1)B(1, n + 2) \\ + a_1^3a_2n(n + 1)B(2, n + 1) + a_1^5n(n + 1)(n + 2) \end{bmatrix} u(n,1,2) ,$$

and

$$N = \begin{bmatrix} a_2^3B(1,n)B(2,n)B(3,n) + a_1^2a_2^2n(n + 1)B(3,n) \\ + a_1^2a_2^2n(n + 1)B(1, n + 2) + a_1^2a_2^2n(n + 1)B(2,n + 1) \\ + a_1^4a_2n(n + 1)(n + 2) \end{bmatrix} u(n - 1,1,2) .$$

It should be noted that the method used in [1] to derive the formulas (8), (8.1), and (8.2) may also be used to develop formulas of the $u(n,k,2)$ for values of $k = 5$ and higher.

In this paper we find for the first time a general method to express the $u(n,k,m)$ as a function of the $u(j,1,m)$ $(j = 0, 1, 2, \cdots)$ with $m \geq 2$ $(m = 2, 3, 4, \cdots)$.

Let

(9) $$y = 1 + \sum_{v=1}^{m} a_v x^v , \quad z = \sum_{v=0}^{m-2} d_v x^v, \quad \text{and} \quad w = \sum_{v=0}^{m-1} b_v x^v ,$$

where a, d and b are rational integers, $m \geq 2$ $(m = 2, 3, \cdots)$ and

(9.1) $$M(m) = zy - w(dy/dx) \qquad (M(m) \text{ is a rational number}) .$$

Now, differentiating the identity $y^{-k} = y^{-k}$, we have

(10) $$-k(dy/dx)/y^{k+1} = d\phi(x)^k/dx ,$$

where

$$\phi(x)^k = \left(\sum_{n=0}^{\infty} u(n,k,m)x^n\right)^k = y^{-k}, \quad k = 1, 2, 3, \cdots, \text{ and } m \geq 2 .$$

We then respectively, multiply (9.1) through by k and divide (9.1) through by y^{k+1} and combine the result with (10). This leads to

$$(kM(m) - kzy)/y^{k+1} = (d\phi(x)^k/dx)w ,$$

and we have

(11) $$xkM(m)/y^{k+1} = xkz/y^k + wx(d\phi(x)^k/dx) .$$

Now, comparing coefficients in (11), we conclude that

(12) $$u(n-1), k+1, m)kM(m) = k\sum_{v=0}^{m-2} u(n-1-v, k, m)d_v + \sum_{v=0}^{m-1} u(n+v+1-m, k, m)(n+v+1-m)b_{m-v-1} .$$

To complete (12), we notice it is necessary to solve (9.1), and this is easily accomplished by collecting the coefficients of x^n. Comparing the coefficients then leads to the following $2m - 1$ equations: (Note: In what follows $B_j = ja_j$, and also for convenience we have replaced a_v with $-a_v$ $(j, v = 1, 2, 3, \cdots, m.)$

(13)

$d_0 = M(m) + B_1b_0 ,$

$a_1d_0 = d_1 + B_2b_0 + B_1b_1 ,$

$a_2d_0 = -a_1d_1 + d_2 + B_3b_0 + B_2b_1 + B_1b_2 ,$

. . . .

$a_{m-2}d_0 = -a_{m-3}d_1 - a_{m-4}d_2 - \cdots - a_1d_{m-3} + d_{m-2} + B_{m-1}b_0 + B_{m-2}b_1 + \cdots + B_1b_{m-2} ,$

$a_{m-1}d_0 = -a_{m-2}d_1 - \cdots - a_2d_{m-3} - a_1d_{m-2} + B_mb_0 + B_{m-1}b_1 + \cdots + B_1b_{m-1} ,$

$a_md_0 = -a_{m-1}d_1 - \cdots - a_3d_{m-3} - a_2d_{m-2} + B_mb_1 + \cdots + B_2b_{m-1} ,$

$0 = -a_md_1 - \cdots - a_4d_{m-3} - a_3d_{m-2} + B_mb_2 + \cdots + B_3b_{m-1} ,$

. . . .

$0 = -a_md_{m-3} - a_{m-1}d_{m-2} + B_mb_{m-2} + B_{m-1}b_{m-1} ,$

$0 = -a_md_{m-2} + B_mb_{m-1}$

(dividing through by a_m this last equation becomes $0 = -d_{m-2} + mb_{m-1}$).

Next we consider in (13) the $2m - 1$ equations in the $2m - 1$ unknowns $M(m), d_1, d_2, \cdots, d_{m-2}, b_0, b_1, \cdots, b_{m-1}$, where for convenience we write

$$\begin{aligned}
&S(m,0,0) = M(m)\,; && S(0,1,0) = d_1, && S(0,2,0) = d_2, \cdots, \\
(14.1)\quad &S(0,\ m-2,\ 0) = d_{m-2}\,; && S(0,0,1) = b_1, && S(0,0,2) = b_2, \cdots, \\
&S(0,0,\ m-1) = b_{m-1}\,; && \text{and} && b_0 = b_0 .
\end{aligned}$$

The $2m - 1$ equations in the $2m - 1$ unknowns $S(g)$ (where we consider g to run through all the $2m - 1$ combinations one at a time of the $S(\ )$ (we also include b_0) in (14.1)) can be solved by Cramer's rule to obtain

$$D(m)S(g) = D(g)\,, \tag{15}$$

where $D(m)$ and $D(g)$ are the determinants given below:

$$(15.1)\qquad D(m) = \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & B_1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & B_2 & B_1 & \cdots & 0 & 0 \\
0 & -a_1 & 1 & \cdots & 0 & 0 & B_3 & B_2 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & & \cdots & \cdot & \cdot \\
0 & -a_{m-3} & -a_{m-4} & \cdots & -a_1 & 1 & B_{m-1} & B_{m-2} & \cdots & B_1 & 0 \\
0 & -a_{m-2} & -a_{m-3} & \cdots & -a_2 & -a_1 & B_m & B_{m-1} & \cdots & B_2 & B_1 \\
0 & -a_{m-1} & -a_{m-2} & \cdots & -a_3 & -a_2 & 0 & B_m & \cdots & B_3 & B_2 \\
0 & -a_m & -a_{m-1} & \cdots & -a_4 & -a_3 & 0 & 0 & \cdots & B_4 & B_3 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & 0 & \cdots & -a_m & -a_{m-1} & 0 & 0 & \cdots & B_m & B_{m-1} \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & m
\end{vmatrix}$$

(Determinant $D(m)$ = the coefficients of the $S(g)$)

and

(15.2) $D(g)$ is the determinant we get when replacing in (15.1) the appropriate column of the coefficients of any $S(g)$ with the column to the extreme left in (13) (the terms in the column to the extreme left in (13) from top to bottom are: $d_0,\ a_1,\ d_0,\ \cdots,\ a_m,\ d_0,\ 0,\ \cdots,\ 0,\ 0$).

<u>Note.</u> Upon investigation we notice that there is no loss of generality if we put

$$d_0 = D(m)\,. \tag{15.3}$$

We shall now use the above method to derive formulas for the generalized Multinacci number.

We first find formulas for the generalized Tribonacci number. We write the generalized Tribonacci power series as follows:

(16) $$(1 - a_1x - a_2x^2 - a_3x^3)^{-k} = \sum_{n=0}^{\infty} u(n,k,3)x^n ,$$

where $k = 1, 2, 3, \cdots$, the a are integers and $u(0,k,3) = 1$.

Now combining (16) with (9.1), we write

(17) $$M(3) = (d_0 + d_1x)(1 - a_1x - a_2x^2 - a_3x^3) + (a_1 + 2a_2 + 3a_3x^3)(b_0 + b_1x + b_2x^2)$$

and combining (17) with (15.1 and 15.3, with $m = 3$), we have

(17.1) $$d_0 = D(3) = \begin{vmatrix} 1 & 0 & B_1 & 0 & 0 \\ 0 & 1 & B_2 & B_1 & 0 \\ 0 & -a_1 & B_3 & B_2 & B_1 \\ 0 & -a_2 & 0 & B_3 & B_2 \\ 0 & -1 & 0 & 0 & 3 \end{vmatrix}$$

and of course applying the directions in (15.2, with $m = 3$) in combination with the determinant $D(3)$ in (17.1), leads to the following:

(17.2) $$\begin{aligned} d_0 &= D(3) = 27a_3^2 + 15a_1a_2a_3 - 4a_2^3 \\ d_1 &= 18a_1a_3^2 - 6a_2^2a_3 \\ b_0 &= 4a_1^2a_3 + 3a_2a_3 - a_1a_2^2 \\ b_1 &= 9a_3^2 + 7a_1a_2a_3 - 2a_2^3 \\ b_2 &= 6a_1a_3^2 - 2a_2^2a_3 \\ M(3) &= 27a_3^2 + 18a_1a_2a_3 + 4a_1^3a_3 - 4a_2^3 - a_1^2a_2^2 . \end{aligned}$$

We now combine (16) and (17.2) with (12, with $m = 3$), which leads to

(18) $$\begin{aligned} &k(27a_3^2 + 18a_1a_2a_3 + 4a_1^3a_3 - 4a_2^3 - a_1^2a_2^2)u(n-1, k+1, 3) \\ &\quad = (4a_1^2a_3 + 3a_2a_3 - a_1a_2^2)nu(n,k,3) \\ &\quad\quad + ((n-1)(9a_3^2 + 7a_1a_2a_3 - 2a_2^3) + k(27a_3^2 + 15a_1a_2a_3 - 4a_2^3))u(n-1, k, 3) \\ &\quad\quad + ((n-2)(6a_1a_3^2 - 2a_2^2a_3) + k(18a_1a_3^2 - 6a_2^2a_3))u(n-2, k, 3) . \end{aligned}$$

(18.1) In (18) it is evident that if we put $k = 1$ we can find the $u(n,2,3)$ as a function of the $u(n,1,3)$ and also for $k = 2$ we find $u(n,3,3)$ as a function of the $u(n,2,3)$, so that we have $u(n,3,3)$ as a function of the $u(n,1,2)$. In this way, step by step for $k > 1$ (with induction added), it is easy to see that we can find formulas of the $u(n,k,3)$ as a function of the $u(n,1,3)$.

(19) Using the exact methods which lead to (18) and (18.1), we find formulas for the Quatronacci $(u(n,k,4))$ numbers (with $k > 1$) as a function of the $u(n,1,4)$, and we find formulas for the generalized Multinacci $(u(n,k,m)$ with $m = 5, 6, 7, \cdots$ and $k > 1)$ numbers as a function of the $u(n,1,m)$.

3. THE GENERALIZED MULTINACCI NUMBER EXPRESSED AS A LIMIT

Note. In [1] the generalized Fibonacci number is expressed as the following:

(20) $$\lim_{n \to \infty} (u(n,\ k+1,\ 2)/(n+1)^k u(n,1,2)) = (1 + a_1(a_1^2 + 4a_2)^{-\frac{1}{2}})^k / 2^k k!\ ,$$

where

$$k,n = 1, 2, 3, \cdots .$$

In this paper we find asymptotic formulas of the $u(n,k,m)$ (with $k,m \geq 2$) expressed in terms of $u(n,1,m)$, a_v, n, and k.

However, before finding our asymptotic formulas, we make some

(21) SUPPLEMENTARY REMARKS

This author, for the first time, proved the following in 1969 [2]. Define

$$\sum_{w=0}^{f} b_w x^w = F(x) \neq 0$$

(for a finite f),

$$\sum_{w=0}^{t} a_w x^w = \prod_{w=1}^{m} (1 - r_w x)^{d_w} = Q(x)$$

for a finite t and m, where the $d_w \neq 0$ are positive integers, the $r_w \neq 0$ and are distinct and we say $|r_1|$ is the greatest $|r|$ in the $|r_w|$. We then proved the following

Theorem. If

$$F(x)/Q(x) = \sum_{w=0}^{\infty} u_w x^w ,$$

then

$$\lim_{n \to \infty} \left| u_n / u_{n-j} \right|$$

(for a finite $j = 0, 1, 2, \cdots$) converges to $|r_1^j|$, where the $r_w \neq 0$ in $Q(x)$ are distinct with distinct moduli and $|r_1|$ is the greatest $|r|$ in the $|r_w|$.

We are now in a position to discuss the generalized Multinacci number expressed as a limit.

First, we consider when $m = 3$ and we multiply equation (18, with $k = 1$) through by $1/nu(n-1,\ 1,\ 3)$ to get

$$M(3)u(n-1,2,3)/nu(n-1,1,3) = b_0u(n,1,3)/u(n-1,1,3) + ((n-1)b_1 + d_0)u(n-1,1,3)/u(n-1,1,3)n + ((n-2)b_1 + d_1)u(n-2,1,3)/u(n-1,1,3)n \tag{22}$$

(23) In (21) we have $u(n,1,3)/u(n-1,1,3) = r$ where r is the greatest root in

$$x^3 - a_1x^2 - a_2x - a_3 = 0,$$

so that equation (22) may be written as

$$\lim_{n\to\infty} M(3)u(n-1,2,3)/nu(n-1,1,3) = rb_0 + b_1 + b_2/r = \text{(say) } L(3). \tag{23.1}$$

Now, we multiply (18, with $k = 2$) through by

$$M(3)/n^2u(n-1,1,3),$$

to get

$$2(M(3))^2u(n-1,3,3)/n^2u(n-1,1,3) =$$
$$+ \left[u(n,2,3)M(3)b_0/nu(n-1,1,3)\right]\left[u(n,1,3)/u(n,1,3)\right]$$
$$+ ((n-1)b_1 + 2d_0)u(n-1,2,3)/n^2u(n-1,1,3)$$
$$+ \left[((n-2)b_2 + 2d_1)u(n-2,2,3)M(3)/n^2u(n-1,1,3)\right]\left[u(n-1,2,3)/u(n-1,2,3)\right],$$

where combining this result with (23.1), and with $n\to\infty$, leads to

$$\lim_{n\to\infty} (2!\,(M(3))^2u(n-1,3,3)/n^2u(n-1,1,3)) = b_0L(3)r + b_2L(3)/r = (b_0r + b_1 + b_2/r)L(3) = (L(3))^2. \tag{24}$$

We continue with the exact method that gave us (24) step by step and with induction, which leads us (for $k = 1, 2, \cdots$) to:

The generalized Tribonacci number expressed as a limit

$$\lim_{n\to\infty} (k!\,(M(3))^ku(n,k+1,3)/(n+1)^ku(n,1,3)) = (L(3))^k, \tag{25}$$

where $L(3)$ is defined in (23.1).

Now, with the exact method that was used in finding (25) applied to the equation in (12) and step by step (and with added induction), we prove that:

The generalized Multinacci number expressed as a limit is

$$\lim_{n\to\infty} (k!\,(M(m))^ku(n,k+1,m)/(n+1)^ku(n,1,m)) = (L(m))^k, \tag{26}$$

where

$$\lim_{n \to \infty} M(m)u(n,2,m)/(n+1)u(n,1,m) = \sum_{v=0}^{m-1} b_v r^{1-v} = \text{(say)}\ L(m) ,$$

r is the greatest root in

$$x^m - \sum_{w=1}^{m} a_w x^{m-w} = 0 ,$$

the $M(m)$ and the b_v are found by using Cramer's rule as defined in (15) through (15.3), $m = 2, 3, 4, \cdots$, $n = 0, 1, 2, \cdots$, $k = 1, 2, 3, \cdots$, and $u(0,k,m) = 1$.

4. A GENERALIZATION OF THE BINOMIAL FORMULA

Put

$$y = \sum_{w=0}^{m} a_w x^w = \sum_{n=0}^{m} a(n,1,m)x^n ,$$

so that

$$y^k = \left(\sum_{w=0}^{m} a_w x^w\right)^k = \sum_{n=0}^{mk} a(n,k,m)\, x^n , \tag{27}$$

where $m = 1, 2, 3, \cdots$, $k = 1, 2, 3, \cdots$, and the a_w are arbitrary numbers $(a_0 \neq 0)$.

It is evident that $y^{k-1}y = y^k$, and combining this identity with (27) and then comparing the coefficients, leads to

$$a(mk - q, k, m) = \sum_{v=0}^{m} a(v,1,m)a(mk - q - v, k - 1, m) , \tag{28}$$

where q ranges through the values $q = 0, 1, 2, \cdots, mk - m$, $k = 2, 3, 4, \cdots$, and $m = 1, 2, 3, \cdots$.

Differentiating equation (27) leads to

$$k\left(\sum_{v=0}^{mk-m} a(v, k-1, m)x^v\right)\left(\sum_{v=1}^{m} va(v,1,m)x^v\right) = \sum_{v=1}^{mk} va(v,k,m)x^v ,$$

and comparing the coefficients in this result, we have

$$(29)\qquad (mk - q)a(mk - q, k, m) = k\sum_{v=1}^{m} va(v,1,m)a(mk - q - v, k - 1, m)\ ,$$

where q ranges through the values $q = 0, 1, 2, \cdots, mk - m$, $k = 2, 3, 4, \cdots$, and $m = 1, 2, 3, \cdots$.

We multiply equation (28) through by $mk - q$ so that the right side of (28) is now an identity with the right side of (29), and arranging the terms in this result leads to

$$(30)\qquad \begin{aligned}&(mk - q)a(0,1,m)a(mk - q, k - 1, m)\\ &\quad = \sum_{v=1}^{m} a(v,1,m)a(mk - q - v, k - 1,m)(vk - mk + q)\ .\end{aligned}$$

Then replacing k with k + 1 in (30), we have

$$(31)\qquad \begin{aligned}&(mk + m - q)a(0,1,m)a(mk + m - q, k, m)\\ &\quad \sum_{v=1}^{m} a(v,1,m)a(mk + k - q - v, k, m)((v - m)(k + 1) + q)\ ,\end{aligned}$$

where $m,k = 1, 2, 3, \cdots$, q ranges through the values $q = 0, 1, 2, \cdots, mk$, $mk + k - q = v \geq 0$, and it is evident that

$$a(0,k,m) = (a(0,1,m))^k, \qquad \text{and} \qquad a(mk,k,m) = (a(m,1,m))^k\ .$$

As an application of (30) we find a value for $a(1,k,m)$. Let $mk + m - q = 1$, so that

$$a(0,1,m)a(1,k,m) = \sum_{v=1}^{m} a(v,1,m)a(1 - v, k, m)(vk + v - 1)\ ,$$

then

$$a(0,1,m)a(1,k,m) = ka(0,k,m)a(1,1,m) = k(a(0,1,m))^k a(1,1,m)$$

and we have

$$a(1,k,m) = k(a(0,1,m))^{k-1}a(1,1,m)\ .$$

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