# A NEW LOOK AT FIBONACCI GENERALIZATION

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## 1. INTRODUCTION

Our parent topic will be sequences, in the broadest sense. That is to say, we shall be dealing with ordered infinite sets of numbers, mostly or usually positive integers, whose character is determined by (a) some given subsequence of s members, and (b) a function linking any given member to its immediate preceding s - ad. In this context the case of s = 1 is trivial, whereas the case of s = 2 includes many well known examples, in particular those called the Fibonacci and Lucas sequences. Some of the examples of the case s = 3 have been discussed under the name of Tribonacci sequences.

Here we restrict attention to s = 2. In characterizing such sequences we use the letters A and B to denote the given pair (and only coprime A and B will be admitted). The determining function will be linear, with parameters N and M. Thus the term following B will be NA + MB; the next NB + M(NA + MB); and so on. Similarly, the term preceding A will be (B - MA)/N; and the next $A/N - M(B - MA)/N^2$; and so on. Each term is in fact expressible as aA + bB, where the coefficients a and b are polynomials in N and M, and if we work through the algebra the results shown in Table 1 will be reached.

Note that we have not so far mentioned ordinal numbers associated with the terms of the sequence. In thinking of the formal sequence, extending to infinity in both directions, we have to realize that there is an arbitrariness in putting ordinals in one-to-one correspondence with the terms. But it is patently convenient to associate the term A with "first," so that all terms less than A are associated with nonpositive ordinals. Not the least reason for this choice is that the structure of the sequence is such that the expression for terms smaller than A is different from, and more complicated than the expression for terms greater than B (the former involve alternating algebraic signs).

Examining Table 1 we observe that it contains the apices of Pascal Triangles, and it is not difficult to show that, with the proposed ordinal convention, the $n^{th}$ term is

$$(1)\qquad \sum_{i=0}^{\infty} \left( \binom{n-i-2}{i-1} MA + \binom{n-i-2}{i} B \right) N^i M^{n-2i-2} \qquad (n > 2)$$

and

$$(2)\qquad (-1)^{n+1} \sum_{m=0}^{\infty} \left( \binom{-n-i+1}{i} MA - \binom{-n-i}{i} B \right) N^{n+i-1} M^{-n-2i} \qquad (n < 1)\,.$$

Table 1
POLYNOMIALS IN N AND M SPECIFYING THE SEQUENCE
(ONE TERM PER LINE) OF [aA + bB], WHERE a = f(N, M) AND b = f'(N, M)

| a = Coefficient of A | b = Coefficient of B |
|---|---|
| ⋮ | ⋮ |
| $-(N^{-5}M^5 + 4N^{-4}M^3 + 3N^{-3}M)$ | $N^{-5}M^4 + 3N^{-4}M^2 + N^{-3}$ |
| $N^{-4}M^4 + 3N^{-3}M^2 + N^{-2}$ | $-(N^{-4}M^3 + 2N^{-3}M)$ |
| $-(N^{-3}M^3 + 2N^{-2}M)$ | $N^{-3}M^2 + N^{-2}$ |
| $N^{-2}M^2 + N^{-1}$ | $-(N^{-2}M)$ |
| $-(N^{-1}M)$ | $N^{-1}$ |
| 1 | 0 |
| 0 | 1 |
| N | M |
| NM | $M^2 + N$ |
| $NM^2 + N^2$ | $M^3 + 2NM$ |
| $NM^3 + 2N^2M$ | $M^4 + 3NM^2 + N^2$ |
| $NM^4 + 3N^2M^2 + N^3$ | $M^5 + 4NM^3 + 3N^2M$ |
| $NM^5 + 4N^2M^3 + 3N^3M$ | $M^6 + 5NM^4 + 6N^2M^2 + N^3$ |

## 2. A TWO-PARAMETER SEQUENCE

In what follows, we shall concentrate on an important special case of the "$\underline{s} = 2$" linear sequences, namely, that with A = M = 1. The setting of A at unity is actually less of a restriction than at first appears, in that any sequence with $A \neq 1$ can be transformed to the "unity" set by division of every term by A. This new sequence will retain most of the properties of its original form, with the notable exception of number-theoretic properties. The setting of M at unity not only introduces a major simplification into the structure, but, as we shall see later, it ties in with a natural extension of the classic Fibonacci Rabbit Problem.

Let us fix a notation at this point. We shall use $F_{B,N,n}$ to denote the $n^{th}$ member of the sequence whose parameters are B ($\geq 0$) and N ($\geq 1$). Thus

$$\begin{cases} F_{B,N,1} = 1; \quad F_{B,N,2} = B \\ F_{B,N,n} = NF_{B,N,n-2} + F_{B,N,n-1} \end{cases} \tag{3}$$

Normally, B and N will be integers. The case of N being any real number $>-1/4$ is worth special consideration; it yields monotonically increasing sequences many of whose properties are shared with those of N integral; but it will not be explored here. Furthermore, we shall not be specifically concerned with $\underline{n}$ negative (although it will occasionally have to be referred to in explication of certain formulas).

The generating function of the sequence is worth noting here. It is the left-hand side of the identity

$$\frac{1 + x(B - 1)/N}{N - x - x^2} = \sum_{n=1}^{\infty} F_{B,N,n} N^{-n} x^{n-1} \quad . \tag{4}$$

This can be verified by multiplying out. And setting $B = N = 1$ we of course obtain the familiar generating function of the "original" Fibonacci sequence, which is $1/(1 - x - x^2)$.

We shall use $\{B, N\}$ to denote the sequence itself, and it must be pointed out at once that not all $\{B, N\}$ are unique, sequence-wise. Some may differ only in "key," to borrow the musical term, in the sense that a shift in the ordinals (the $\underline{n}$-sequence) will make them identical. For example, the following three sequences can be equalized by such shifts:

| n : | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|
| $\{0,1\}$ : | 5 | -3 | 2 | -1 | 1 | 0 | 1 | 1 |
| $\{1,1\}$ : | 2 | -1 | 1 | 0 | 1 | 1 | 2 | 3 |
| $\{2,1\}$ : | -1 | 1 | 0 | 1 | 1 | 2 | 3 | 5 |

Explanation is superfluous.

Another type of hidden identity (for the segments with $\underline{n} > 0$) is multiplicative, and is illustrated below:

| n : | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $\{0,3\}$ : | 4/9 | -1/3 | 1 | 0 | 3 | 3 | 12 |
| $\{0,3\}/3$ : | 4/27 | -1/9 | 1/3 | 0 | 1 | 1 | 4 |
| $\{1,3\}$ : | 1/3 | 0 | 1 | 1 | 4 | 7 | 19 |

Thus $\{0, N\}$, divided throughout by N is identical, over positive $\underline{n}$ (apart from a 2n-keyshift), to $\{1, N\}$.

Using a subscript to denote keyshift, we can summarize the algebra of these sequences as follows:

$$\{0, Y\} = Y\{1, Y\}_{+2} = Y\{Y + 1, Y\}_{+3} \tag{5}$$

which of course includes the special case of $Y = 1$, illustrated above. Furthermore, if $B|N$, then

(6) $$\{X, Y\} = X\{Y/X + 1, Y\}_{+1}$$

which has a special case $X = Y$, so that

(7) $$\{X, Y\} = X\{2, Y\} = 2X\{Y/2 + 1, Y\}_{+2} \qquad (Y \text{ even})$$

And if $Y = X(X - 1)$, the sequence is simply the powers of $X$, and is infinitely divisible by $X$ — but every quotient is identical to the original dividend, apart from a shift of key. Symbolically,

(8) $$F_{X(X-1), X, n} = X^{n-1} \qquad (X \geq 1) \ .$$

Finally, if $X > Y + 1$, all $\{X, Y\}$ are unique.

In Figure 1, the distribution pattern of these hidden identities is shown for some of the lower B and N. Each cell is to be regarded as containing a complete sequence $\{B, N\}$ — specifically, $\{X, Y\}$. A blank cell is understood to contain an irreducible sequence (in the sense that it cannot be transformed, by division and/or shift of key, into a smaller-B sequence). Hatched cells contain sequences that are powers of B. Black cells hold all other reducible sequences.

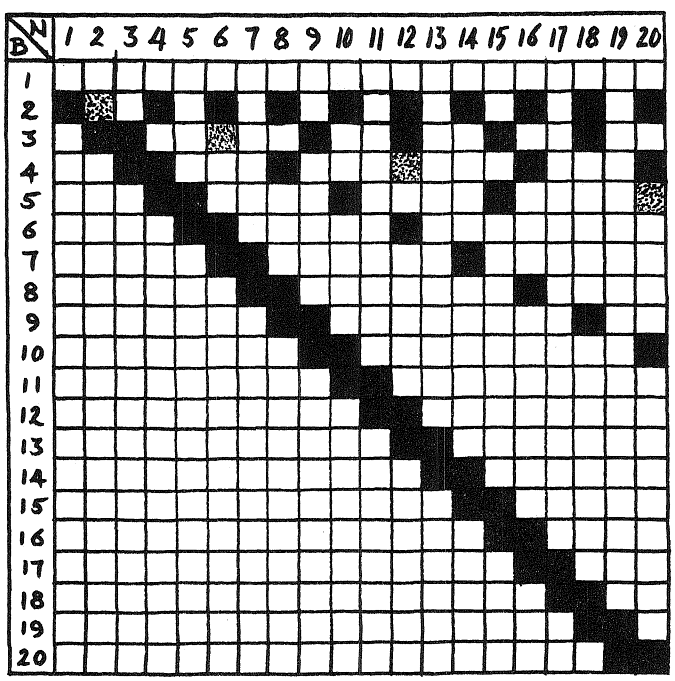


Fig. 1 Distribution of Three Types of $\{B, N\}$: (i) reducible (black); (ii) powers of B (stippled); (ii) irreducibles.

In the Appendix are collected for reference $F_{B,N,n}$ for $n = 1(1)25$, and for certain B ($\not>5$) and N ($\not>10$). Of the possible total of 50 combinations of B and N, only 34 have been tabulated: 14 were omitted because of their being reducibles, and 2 because of their being merely sequences of powers (which in this context are uninteresting). The omissions, in short, are conditioned by Fig. 1.

## 3. PROLIFIC FIBRABBITS

The sequence $\{1,1\}$ is the original Fibonacci sequence, and $\{3,1\}$ is the Lucas sequence — and we can now see why the Lucas sequence is normally regarded as the one "next" to the Fibonacci sequence; it is because the intervening $\{2,1\}$ is really $\{1,1\}$, with a unit shift of key. We may note in passing that (for any given fixed N, say Y) the identity

$$F_{1,Y,n} + F_{1,Y,n-2} = F_{Y+2,Y,n-1} \tag{9}$$

yields the well known relation between member of the Fibonacci and Lucas sequences when we set Y at unity.

The interesting thing about $\{1,N\}$ is that it furnishes solutions to the Fibonacci Rabbit problem generalized to the situation in which each pair gives birth to N pairs at a time, instead of one. This is perhaps best appreciated by reference to a time-table, as in Table 2.

Table 2

NUMBER OF PAIRS OF IMMORTAL RABBITS ALIVE,
BY MONTH (t) AND GENERATION (g in $N^g$),
IN A BREEDING REGIME THAT UNFAILINGLY YIELDS N PAIRS PER MONTHLY BIRTH

| t = n - 1 | $N^0$ | $N^1$ | $N^2$ | $N^3$ | $N^4$ | $N^5$ | $N^6$ | Sum when N = 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | 1 | 1 | 1 |
| 1 | 1 | | | | | | | 1 | 1 | 1 |
| 2 | 1 | 1 | | | | | | 2 | 3 | 4 |
| 3 | 1 | 2 | | | | | | 3 | 5 | 7 |
| 4 | 1 | 3 | 1 | | | | | 5 | 11 | 19 |
| 5 | 1 | 4 | 3 | | | | | 8 | 21 | 40 |
| 6 | 1 | 5 | 6 | 1 | | | | 13 | 43 | 97 |
| 7 | 1 | 6 | 10 | 4 | | | | 21 | 85 | 217 |
| 8 | 1 | 7 | 15 | 10 | 1 | | | 34 | 171 | 508 |
| 9 | 1 | 8 | 21 | 20 | 5 | | | 55 | 341 | 1159 |
| 10 | 1 | 9 | 28 | 35 | 15 | 1 | | 89 | 683 | 2683 |
| 11 | 1 | 10 | 36 | 56 | 35 | 6 | | 144 | 1365 | 6160 |
| 12 | 1 | 11 | 45 | 84 | 70 | 21 | 1 | 233 | 2731 | 14209 |
| 13 | 1 | 12 | 55 | 120 | 126 | 56 | 7 | 377 | 5461 | 32689 |

We imagine, after Fibonacci, a pair of month-old rabbits mated in an enclosure, and giving birth to N new pairs every month thereafter; and each of the new pairs breeds similarly after a month's maturation. The table can be readily constructed from elementary considerations, with each column representing a generation, beginning at the zeroth — and the construction is in fact the familiar tilted Pascal Triangle. At the beginning of the second month there will be $1 + N$ pairs; at the beginning of the third there will be $1 + 2N$ pairs, and so forth. Clearly, the sums in the end columns will be

$$\sum_{i=0}^{\infty} \binom{n-i-1}{i} N^i \tag{10}$$

— which is expression (1) with $A = B = M = 1$ and the utilization of Pascal's Rule for the addition of binomial coefficients. In other words, Eq. (10) is $F_{1,N,n}$.

It is possible to sophisticate the treatment by allowance for deaths, the simplest situation being to schedule the death of a mated pair of rabbits immediately after the birth of its $m^{th}$ litter. Hoggatt and Lind [2] have shown how this can be done for the classic case, in which $N = 1$. For $N > 1$ the crude arithmetic of the population growth is straightforward enough, but it does not condense well. The population increment from the $g^{th}$ generation at time $\underline{n}$ $(= t + 1)$ can be written

$$\sum_{i=0}^{\infty} (-1)^i \binom{g-1}{i} \left( \binom{h}{g-1} - \binom{h-m}{g-1} - \binom{h-m-1}{g-1} + \binom{h-2m-1}{g-1} \right) , \tag{11}$$

where

$$h = n - g - im - 2 ,$$

and the summation of (11) over all $\underline{g}$ and all time points to $\underline{n}$ gives the required population size at $\underline{n}$. This is clumsy, but a compact operation is elusive.

Actually, allowance for restricted littering and for mortality does not make a great difference to the population, which, with $N > 1$, soon becomes enormous. For example, $F_{1,5,23} = 3\,912\,125\,981$, and if we limit $\underline{m}$ to 5 (and remove the parents subsequently), the population at the $23^{rd}$ month will still be 3 759 051 250, which is 96 percent of the former figure (and represents more than one pair of rabbits for every human being on earth).

Incidentally, in considering litters with more than two siblings, we can easily cope with a sex ratio other than 50:50. Suppose, for instance, that litters of five bucks and four does are to be substituted for the classic one buck and one doe (per litter): we carry out the arithmetic for $N = 4$, and then multiply the answer by the factor $(4 + 5)/4$; this will give us the required population (in, of course, rabbits, not pairs of rabbits).

## 4. $\tau_N$ AND THE EXPLICIT FORMULAS

A sequence of the kind we are discussing may intuitively be expected to have a limiting ratio of adjacent terms, and in fact it is well established that such a ratio exists and is independent of B. But it is not independent of N. By extension from the familiar treatment of the case of $\{1,1\}$, we write the auxiliary equation

$$\tau_N^n = N\tau_N^{n-2} + \tau_N^{n-1} \tag{12}$$

and divide it by $\tau_N^{n-2}$ to give, after rearrangement,

$$\tau_N^2 - \tau_N - N = 0 \; . \tag{13}$$

The roots of (13) are $1/2 \pm \sqrt{N + 1/4}$, and we identify the positive root with the required limiting ratio, $\tau_N$. The other root, we note, is $1 - \tau_N$.

So the asymptotic growth rate (per unit interval) of all $\{B,1\}$ (including the original Fibonacci and Lucas sequences) is $1/2 + \sqrt{5}/2 = 1.618034\cdots$; that of all $\{B,2\}$ is $1/2 + \sqrt{9}/2$; that of all $\{B,3\}$ is $1/2 + \sqrt{13}/2 = 2.302775\cdots$; and so on. These asymptotes are approached rapidly: turning to the sums at the right foot of Table 2, for example, we shall find that $377/233 = 1.618\cdots$, that $5461/2731 = 2.000\cdots$, and that $32689/14209 = 2.301\cdots$.

The powers of $\tau_N$ can be expressed in terms of two F's, thus:

$$\tau_N^n = \frac{F_{1+2X,X,n} + F_{1,N,n}\sqrt{4N+1}}{2} \tag{14}$$

and

$$\tau_N^{-n} = \frac{F_{1+2X,X,n} - F_{1,N,n}\sqrt{4N+1}}{2}\,(-1/N)^n \; , \tag{15}$$

where X is the particular value of N and determines B in the first F of the numerator.

The quantity $\tau_N$ can be used to derive explicit expressions for any $F_{B,N,n}$ by virtue of the relation

$$F_{B,N,n} = k_1\tau_N^{n-1} + k_2(1 - \tau_N)^{n-1} \; , \tag{16}$$

where the k's are constants that can be evaluated from our knowledge of the two parametric members of the sequence

$$F_{B,N,1} = 1 = k_1 + k_2$$

and

$$F_{B,N,2} = B = k_1\tau_N + k_2(1 - \tau_N) \; ,$$

whence

$$k_1 = (\tau_N + B - 1)/(2\tau_N - 1)$$

$$k_2 = (\tau_N - B)/(2\tau_N - 1) \; .$$

Therefore,

$$F_{B,N,n} = \frac{(\tau_N + B - 1)\tau_N^{n-1} + (\tau_N - B)(1 - \tau_N)^{n-1}}{2\tau_N - 1}$$

(17)

$$= \frac{(\tau_N + B - 1)(\tau_N - 1)\tau_N^n - (\tau_N - B)\tau_N(1 - \tau_N)^n}{N(2\tau_N - 1)}$$

(because $\tau_N(\tau_N - 1) = N$).

It is perhaps worthwhile recasting (17) without $\tau_N$. In so doing we write $\sqrt{N + 1/4} = R$, and obtain

(18) $$F_{B,N,n} = \frac{[N - (B-1)(1/2 - R)](1/2 + r)^n - [N - (B-1)(1/2 + R)](1/2 - R)^n}{2NR} .$$

It is here to be noted that, in particular,

(19) $$F_{1,N,n} = \frac{(1/2 + R)^n - (1/2 - R)^n}{2R} ,$$

which, with $N = 1$, yields the established explicit formula for a member of the original Fibonacci sequence. And, again,

(20) $$F_{3,N,n} = (1/2 + R)^n + (1/2 - R)^n ,$$

which, with $N = 1$, yields the established explicit formula for a member of the Lucas sequence.

## 5. SOME IDENTITIES

Our topic is rich in interesting identities, and in this section a few of the more important ones will be set out together with their degeneralizations to more familiar forms. We omit proofs, which can be constructed on traditional (and mostly inductional) lines — many exercises and problems can in fact be drawn from the statements.

One of the simplest and most revealing of the identities, an almost obvious consequence of expression (1), is

(21) $$F_{B,N,n} = NF_{1,N,n-2} + BF_{1,N,n-1} .$$

An allied identity is

(22) $$F_{B,N,n} = XF_{1,N,n-1} + F_{B-X,N,n}$$

with the special case in which $X = B - 1$:

(23) $$F_{B,N,n} = (B - 1)F_{1,N,n-1} + F_{1,N,n} .$$

Summations of terms and powers of terms are often neatly expressible. For example:

$$\sum_{i=1}^{n} F_{B,N,i} = (F_{B,N,n+2} - B)/N \tag{24}$$

and its relation to the familiar $\{1,1\}$ is plain to see.

The sum of squares to a given $\underline{n}$ can be eompactly expressed for $N = 1$:

$$\sum_{i=1}^{n} F^2_{B,1,i} = F_{B,1,n}F_{B,1,n+1} - (B - 1) \tag{25}$$

but less so for $B = 1$:

$$\sum_{i=1}^{n} F^2_{1,N,i} = \frac{N^3F^2_{1,N,n-1} + N(N^2 - N - 1)F^2_{1,N,n} - F^2_{1,N,n+1} - (N - 1)}{N(N + 1)(N - 2)} \tag{26}$$

which, with $N = 1$, becomes

$$= (F^2_{1,1,n+1} + F^2_{1,1,n} - F^2_{1,1,n-1})/2 = F_{1,1,n}F_{1,1,n+1} \; .$$

A central identity, with several useful reductions, is

$$F_{B,N,n}F_{B,N,n+x+y} - F_{B,N,n+x}F_{B,N,n+y} = (-1)^n N^{n-1} F_{1,N,x}F_{1,N,y}(B^2 - B - N) \; . \tag{27}$$

Setting $y = -x$, and bearing in mind that $F_{1,N,-n} = (-1)^{n-1}N^n F_{1,N,n}$, we can reduce (27) to

$$F^2_{B,N,n} - F_{B,N,n-x}F_{B,N,n+x} = (-1)^{n+x-1}N^{n-x-1}F^2_{1,N,x}(B^2 - B - N) \; . \tag{28}$$

And setting $x = -y = 1$ gives us

$$F^2_{B,N,n} - F_{B,N,n-1}F_{B,N,n+1} = (-1)^n N^{n-2}(B^2 - B - N) \; . \tag{29}$$

Lastly, as regards reduction of (27), if we set $x = y = n' - 1$, and $n = 1$, we obtain (after depriming $n'$):

$$F^2_{B,N,n} - F_{B,N,2n-1} = F^2_{1,N,n-1}(B^2 - B - N) \tag{30}$$

(and this, when $B = N = 1$, becomes the well known two-consecutive-square identity in $\{1,1\}$).

A general "adjacent products" identity is

$$F_{B,N,n+x} = NF_{B,N,n-1}F_{B,N,x} + F_{B,N,n}F_{B,N,x+1} - (B - 1)F_{B,N,n+x-1} \tag{31}$$

which, when $x = n$, can be expressed in several forms:

$$(32)\quad \begin{aligned} F_{B,N,2n} &= F_{B,N,n}(NF_{B,N,n-1} + F_{B,N,n+1}) - (B-1)F_{B,N,2n-1} \\ &= F^2_{B,N,n+1} - N^2F^2_{B,N,n-1} - (B-1)F_{B,N,2n-1} \\ &= 2F_{B,N,n}F_{B,N,n+1} - F^2_{B,N,n} - (B-1)F_{B,N,2n-1} \end{aligned}$$

(and from the first of which we readily infer that iff $B = 1$, then $F_{B,N,2n}$ must be composite (being divisible by $F_{B,N,n}$)).

If, in (31), we put $x = 2n$, the result is

$$(33)\quad F_{B,N,3n} = NF_{B,N,n-1}F_{B,N,2n} + F_{B,N,n}F_{B,N,2n+1} - (B-1)F_{B,N,3n-3} .$$

And here are two cubic relations that apply when $B$ is unity:

$$(34)\quad \begin{aligned} F_{1,N,3n} &= 3NF_{1,N,n-1}F_{1,N,n}F_{1,N,n+1} + (N+1)F^3_{1,N,n} \\ &= F^3_{1,N,n+1} + NF^3_{1,N,n} - N^3F^3_{1,N,n-1} \end{aligned}$$

— the former of which, incidentally, tells us that $F_{1,N,0}$ (mod 3) is always composite.

## 6. SOME MISCELLANEOUS POINTS

1. In Section 2, it is mentioned that real $N < -1/4$ is out of court, so to say. The reason is that the discriminant of the roots of the generalized Fibonacci quadratic is zero at $N = -1/4$, and negative beyond. At $N = -1/4$ we have that $F_{1,N,n} = n/2^{n-1}$, so that

$$\tau_N = [\lim,\ n \to \infty](n+1)/2n = 1/2 .$$

At $N < -1/4$ the terms of the sequence take alternating algebraic signs, and there is no limiting ratio in the usual sense; what happens of course is that $\tau_N$ moves onto the gaussian plane.

2. The number-theoretic properties of $\{B,N\}$ need examination. It seems clear that the main theorems of divisibility and primality [3] applicable to $\{1,1\}$ also apply, mutatis mutandis, to $\{1,N\}$. And squares are rare among the F's in the Appendix (outside of $\{1,1\}$, in which it is known that only $F_{1,1,12}$ is a square, and beyond $F_{B,N,4}$) I find only $F_{1,4,8} = 441$, and $F_{1,8,6} = 225$. (Note that $X(X-1), X$, which is a sequence of powers, contains an infinity of squares, but this is an oddity.)

Interesting problems in this area take the form: In how many ways, if at all, can a given natural number be represented as $F_{B,N,n}$?

3. The digits of a Fibonacci number, at a given decimal place, occur in cycles along the ascending sequence. Lagrange, says Coxeter [1], observed that the final digits of $\{1,1\}$

repeat in cycles of 60. The question naturally arises as to the cycling pattern of other $\{B, N\}$. The answer is in Table 3.

Table 3
CYCLE SIZE OF REPEATED FINAL DIGITS IN $\{B, N\}$ (EXCLUDING $F_{B,N,1}$)

| B mod 5 \ N mod 10 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0, 1, and 2 | 1 | 60 | 4 | 24 | 6 | 3 | 20 | 12 | 24 | 6 |
| 3 | 1 | 12 | 4 | 24 | 6 | 3 | 4 | 12 | 24 | 6 |
| 4 | 1 | 60 | 2 | 24 | 6 | 3 | 20 | 6 | 24 | 6 |

## REFERENCES


1. H. S. M. Coxeter, Introduction to Geometry, Wiley, New York, 1967, p. 168.
2. V. E. Hoggatt, Jr., and D. A. Lind, "The Dying Rabbit Problem," Fibonacci Quarterly, Vol. 7, No. 4 (1969), pp. 482-487.
3. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Company, New York, 1961.


## APPENDIX

VARIOUS $F_{B,N,n}$ TO $n = 25$

The tables appear on the following pages.

## CONFERENCE PROGRAM
## FIBONACCI ASSOCIATION MEETING

Saturday, October 21, 1972 San Jose State University, Macquarrie Hall

| | |
|---|---|
| 9:15 a.m. | Registration |
| 9:30 - 10:20 | SOME QUASI-EXOTIC THEOREMS<br>Dmitri Thoro, Professor of Mathematics, San Jose State University |
| 10:30 - 11:20 | GENERALIZED LEO MOSER PROBLEMS<br>Pat Gomez, Student, San Jose State University |
| 11:30 - 12:00 | FUN WITH FIBONACCI AT THE CHESS MATCH AND THE BALL PARK<br>Marjorie Bicknell, Mathematics Teacher, A. C. Wilcox High School |
| 1:30 - 2:20 | INTERVALS CONTAINING INFINITELY MANY SETS OF ALGEBRAIC INTEGERS — Raphael Robinson, Professor of Mathematics, University of California, Berkeley |
| 2:30 - 3:20 | SOME ADDITION THEOREMS IN NUMBER THEORY<br>C. T. Long, Professor of Mathematics, Washington State University, Visiting University of British Columbia |
| 3:30 - 4:10 | SOME CONGRUENCES OF THE FIBONACCI NUMBERS MODULO A PRIME,<br>V. E. Hoggatt, Jr., San Jose State University |

LINEARLY GENERALIZED FIBONACCI NUMBERS
$F_{B,N,n}$ WITH B = 1

| n \ N | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 4 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 |
| 5 | 5 | 11 | 19 | 29 | 41 | 55 | 71 | 89 | 109 | 131 |
| 6 | 8 | 21 | 40 | 65 | 96 | 133 | 176 | 225 | 280 | 341 |
| 7 | 13 | 43 | 97 | 181 | 301 | 463 | 673 | 937 | 1 261 | 1 651 |
| 8 | 21 | 85 | 217 | 441 | 781 | 1 261 | 1 905 | 2 737 | 3 781 | 5 061 |
| 9 | 34 | 171 | 508 | 1 165 | 2 286 | 4 039 | 6 616 | 10 233 | 15 130 | 21 571 |
| 10 | 55 | 341 | 1 159 | 2 929 | 6 191 | 11 605 | 19 951 | 32 129 | 49 159 | 72 181 |
| 11 | 89 | 683 | 2 683 | 7 589 | 17 621 | 35 839 | 66 263 | 113 993 | 185 329 | 287 891 |
| 12 | 144 | 1 365 | 6 160 | 19 305 | 48 576 | 105 469 | 205 920 | 371 025 | 627 760 | 1 009 701 |
| 13 | 233 | 2 731 | 14 209 | 49 661 | 136 681 | 320 503 | 669 761 | 1 282 969 | 2 295 721 | 3 888 611 |
| 14 | 377 | 5 461 | 32 689 | 126 881 | 379 561 | 953 317 | 2 111 201 | 4 251 169 | 7 945 561 | 13 985 621 |
| 15 | 610 | 10 923 | 75 316 | 325 525 | 1 062 966 | 2 876 335 | 6 799 528 | 14 514 921 | 28 607 050 | 52 871 731 |
| 16 | 987 | 21 845 | 173 383 | 833 049 | 2 960 771 | 8 596 237 | 21 577 935 | 48 524 273 | 100 117 099 | 192 727 941 |
| 17 | 1 597 | 43 691 | 399 331 | 2 135 149 | 8 275 601 | 25 854 247 | 69 174 631 | 164 643 641 | 357 580 549 | 721 445 251 |
| 18 | 2 584 | 87 381 | 919 480 | 5 467 345 | 23 079 456 | 77 431 669 | 220 220 176 | 552 837 825 | 1 258 634 440 | 2 648 724 661 |
| 19 | 4 181 | 174 763 | 2 117 473 | 14 007 941 | 64 457 461 | 232 557 151 | 704 442 593 | 1 869 986 953 | 4 476 859 381 | 9 863 177 171 |
| 20 | 6 765 | 349 525 | 4 875 913 | 35 877 321 | 179 854 741 | 697 147 165 | 2 245 983 825 | 6 292 689 553 | 15 804 569 341 | 36 350 423 781 |
| 21 | 10 946 | 699 051 | 11 228 332 | 91 909 085 | 502 142 046 | 2 092 490 071 | 7 177 081 976 | 21 252 585 177 | 56 096 303 770 | 134 982 195 491 |
| 22 | 17 711 | 1 398 101 | 25 856 071 | 235 418 369 | 1 401 415 751 | 6 275 373 061 | 22 898 968 751 | 71 594 101 601 | 198 337 427 839 | 498 486 433 301 |
| 23 | 28 657 | 2 796 203 | 59 541 067 | 603 054 709 | 3 912 125 981 | 18 830 313 487 | 73 138 542 583 | 241 614 783 017 | 703 204 161 769 | 1 848 308 388 211 |
| 24 | 46 368 | 5 592 405 | 137 109 280 | 1 544 728 185 | 10 919 204 736 | 56 482 551 853 | 233 431 323 840 | 814 367 595 825 | 2 488 241 012 320 | 6 833 172 721 221 |
| 25 | 75 025 | 11 184 811 | 315 732 481 | 3 956 947 021 | 30 479 834 641 | 169 464 432 775 | 745 401 121 921 | 2 747 285 859 961 | 8 817 078 468 241 | 25 316 256 603 331 |

LINEARLY GENERALIZED FIBONACCI NUMBERS

$F_{B,N,n}$ WITH B = 2

| n \ N | 3 | 5 | 7 | 9 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 |
| 3 | 5 | 7 | 9 | 11 |
| 4 | 11 | 17 | 23 | 29 |
| 5 | 26 | 52 | 86 | 128 |
| 6 | 59 | 137 | 247 | 389 |
| 7 | 137 | 397 | 849 | 1 541 |
| 8 | 314 | 1 082 | 2 578 | 5 042 |
| 9 | 725 | 3 067 | 8 521 | 18 911 |
| 10 | 1 667 | 8 477 | 26 567 | 64 289 |
| 11 | 3 824 | 23 812 | 86 214 | 234 488 |
| 12 | 8 843 | 66 197 | 272 183 | 813 089 |
| 13 | 20 369 | 185 257 | 875 681 | 2 923 481 |
| 14 | 46 898 | 516 242 | 2 780 692 | 10 241 282 |
| 15 | 108 005 | 1 442 527 | 8 910 729 | 36 552 611 |
| 16 | 248 699 | 4 023 737 | 28 377 463 | 128 724 149 |
| 17 | 572 714 | 11 236 372 | 90 752 566 | 457 697 648 |
| 18 | 1 318 811 | 31 355 057 | 289 394 807 | 1 616 214 989 |
| 19 | 3 036 953 | 87 536 917 | 924 662 769 | 5 735 493 821 |
| 20 | 6 993 386 | 244 312 202 | 2 950 426 418 | 20 281 428 722 |
| 21 | 16 104 245 | 681 996 787 | 9 423 065 801 | 71 900 873 111 |
| 22 | 37 084 403 | 1 903 557 797 | 30 076 050 727 | 254 433 731 609 |
| 23 | 85 397 138 | 5 313 541 732 | 96 037 511 334 | 901 541 589 608 |
| 24 | 196 650 347 | 14 831 330 717 | 306 569 866 423 | 3 191 445 174 089 |
| 25 | 452 841 761 | 41 399 039 377 | 978 832 445 761 | 11 305 319 480 561 |

LINEARLY GENERALIZED FIBONACCI NUMBERS

$F_{B,N,n}$ WITH B = 3

| n \ N | 1 | 4 | 5 | 7 | 8 | 10 |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3 | 4 | 7 | 8 | 10 | 11 | 13 |
| 4 | 7 | 19 | 23 | 31 | 35 | 43 |
| 5 | 11 | 47 | 63 | 101 | 123 | 173 |
| 6 | 18 | 123 | 178 | 318 | 403 | 603 |
| 7 | 29 | 311 | 493 | 1 025 | 1 387 | 2 333 |
| 8 | 47 | 803 | 1 383 | 3 251 | 4 611 | 8 363 |
| 9 | 76 | 2 047 | 3 848 | 10 426 | 15 707 | 31 693 |
| 10 | 123 | 5 259 | 10 763 | 33 183 | 52 595 | 115 323 |
| 11 | 199 | 13 447 | 30 003 | 106 165 | 178 251 | 432 253 |
| 12 | 322 | 34 483 | 83 818 | 338 446 | 599 011 | 1 585 483 |
| 13 | 521 | 88 271 | 233 833 | 1 081 601 | 2 025 119 | 5 908 013 |
| 14 | 843 | 226 203 | 652 923 | 3 450 723 | 6 817 107 | 21 762 843 |
| 15 | 1 364 | 579 287 | 1 822 088 | 11 021 930 | 23 017 259 | 80 842 973 |
| 16 | 2 207 | 1 484 099 | 5 086 703 | 35 176 991 | 77 554 115 | 298 471 403 |
| 17 | 3 571 | 3 801 247 | 14 197 143 | 112 330 501 | 261 692 187 | 1 106 901 133 |
| 18 | 5 778 | 9 737 643 | 39 630 658 | 358 569 438 | 882 125 107 | 4 091 615 163 |
| 19 | 9 349 | 24 942 631 | 110 616 373 | 1 144 882 945 | 2 975 662 603 | 15 160 626 493 |
| 20 | 15 127 | 63 893 203 | 308 769 663 | 3 654 869 011 | 10 032 663 459 | 56 076 778 123 |
| 21 | 24 476 | 163 663 727 | 861 851 528 | 11 669 049 626 | 33 837 964 283 | 207 683 043 053 |
| 22 | 39 603 | 419 236 539 | 2 405 699 843 | 37 253 132 703 | 114 099 271 955 | 768 450 824 283 |
| 23 | 64 079 | 1 073 891 447 | 6 714 957 483 | 118 936 480 085 | 384 802 986 219 | 2 845 281 254 813 |
| 24 | 103 682 | 2 750 827 603 | 18 743 456 698 | 379 708 409 006 | 1 297 597 161 859 | 10 529 789 497 643 |
| 25 | 167 761 | 7 046 403 391 | 52 318 244 113 | 1 212 263 769 601 | 4 376 021 251 611 | 38 982 602 045 773 |

LINEARLY GENERALIZED FIBONACCI NUMBERS

$F_{B,N,n}$ WITH B = 4

| n \ N | 1 | 2 | 5 | 6 | 7 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 3 | 5 | 6 | 9 | 10 | 11 | 13 | 14 |
| 4 | 9 | 14 | 29 | 34 | 39 | 49 | 54 |
| 5 | 14 | 26 | 74 | 94 | 116 | 166 | 194 |
| 6 | 23 | 54 | 219 | 298 | 389 | 607 | 734 |
| 7 | 37 | 106 | 589 | 862 | 1 201 | 2 101 | 2 674 |
| 8 | 60 | 214 | 1 684 | 2 650 | 3 924 | 7 564 | 10 014 |
| 9 | 97 | 426 | 4 629 | 7 822 | 12 331 | 26 473 | 36 754 |
| 10 | 157 | 854 | 13 049 | 23 722 | 39 799 | 94 549 | 136 894 |
| 11 | 254 | 1 706 | 36 194 | 70 654 | 126 116 | 332 806 | 504 434 |
| 12 | 411 | 3 414 | 101 439 | 212 986 | 404 709 | 1 183 747 | 1 873 374 |
| 13 | 665 | 6 826 | 282 409 | 636 910 | 1 287 521 | 4 179 001 | 6 917 714 |
| 14 | 1 076 | 13 654 | 789 604 | 1 914 826 | 4 120 484 | 14 832 724 | 25 651 454 |
| 15 | 1 741 | 27 306 | 2 201 649 | 5 736 286 | 13 133 131 | 52 443 733 | 94 828 594 |
| 16 | 2 817 | 54 614 | 6 149 669 | 17 225 242 | 41 976 519 | 185 938 249 | 351 343 134 |
| 17 | 4 558 | 109 226 | 17 157 914 | 51 642 958 | 133 908 436 | 657 931 846 | 1 299 629 074 |
| 18 | 7 375 | 218 454 | 47 906 259 | 154 994 410 | 427 744 079 | 2 331 376 087 | 4 813 060 414 |
| 19 | 11 933 | 436 906 | 133 695 829 | 464 852 158 | 1 365 103 121 | 8 252 762 701 | 17 809 351 154 |
| 20 | 19 308 | 873 814 | 373 227 124 | 1 394 818 618 | 4 359 311 604 | 29 235 147 484 | 65 939 955 294 |
| 21 | 31 241 | 1 747 626 | 1 041 706 269 | 4 183 931 566 | 13 915 033 451 | 103 510 011 793 | 244 033 466 834 |
| 22 | 50 549 | 3 495 254 | 2 907 841 889 | 12 552 843 274 | 44 430 214 679 | 366 626 339 149 | 903 433 019 774 |
| 23 | 81 790 | 6 990 506 | 8 116 373 234 | 37 656 432 670 | 141 835 448 836 | 1 298 216 445 286 | 3 343 767 688 114 |
| 24 | 132 339 | 13 981 014 | 22 655 582 679 | 112 973 492 314 | 452 846 951 589 | 4 597 853 497 627 | 12 378 097 885 584 |
| 25 | 214 129 | 27 962 026 | 63 237 448 849 | 338 912 088 334 | 1 445 695 093 441 | 16 281 801 505 201 | 45 815 774 766 994 |

LINEARLY GENERALIZED FIBONACCI NUMBERS

$F_{B,N,n}$ WITH B = 5

| n \ N | 1 | 2 | 3 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 3 | 6 | 7 | 8 | 11 | 12 | 13 | 14 |
| 4 | 11 | 17 | 23 | 41 | 47 | 53 | 59 |
| 5 | 17 | 31 | 47 | 107 | 131 | 157 | 185 |
| 6 | 28 | 65 | 116 | 353 | 460 | 581 | 716 |
| 7 | 45 | 127 | 257 | 995 | 1 377 | 1 837 | 2 381 |
| 8 | 73 | 257 | 605 | 3 113 | 4 597 | 6 485 | 8 825 |
| 9 | 118 | 511 | 1 376 | 9 083 | 14 236 | 21 181 | 30 254 |
| 10 | 191 | 1 025 | 3 191 | 27 761 | 46 415 | 73 061 | 109 769 |
| 11 | 309 | 2 047 | 7 319 | 82 259 | 146 067 | 242 509 | 381 965 |
| 12 | 500 | 4 097 | 16 892 | 248 825 | 470 972 | 826 997 | 1 369 076 |
| 13 | 809 | 8 191 | 38 849 | 742 379 | 1 493 441 | 2 767 069 | 4 806 761 |
| 14 | 1 309 | 16 385 | 89 525 | 2 235 329 | 4 790 245 | 9 383 045 | 17 128 445 |
| 15 | 2 118 | 32 767 | 206 072 | 6 689 603 | 15 244 332 | 31 519 597 | 60 389 294 |
| 16 | 3 427 | 65 537 | 474 647 | 20 101 577 | 48 776 047 | 106 583 957 | 214 545 299 |
| 17 | 5 545 | 131 071 | 1 092 863 | 60 239 195 | 155 486 371 | 358 740 733 | 758 048 945 |
| 18 | 8 972 | 262 145 | 2 516 804 | 180 848 657 | 496 918 700 | 1 211 412 389 | 2 688 956 636 |
| 19 | 14 517 | 524 287 | 5 795 393 | 542 283 827 | 1 585 323 297 | 4 081 338 253 | 9 511 397 141 |
| 20 | 23 489 | 1 048 577 | 13 345 805 | 1 627 375 769 | 5 063 754 197 | 13 772 637 365 | 33 712 006 865 |
| 21 | 38 006 | 2 097 151 | 30 731 984 | 4 881 078 731 | 16 161 017 276 | 46 423 343 389 | 119 314 581 134 |
| 22 | 61 495 | 4 194 305 | 70 769 399 | 14 645 333 345 | 51 607 296 655 | 156 604 442 309 | 422 722 642 919 |
| 23 | 99 501 | 8 388 607 | 162 965 351 | 43 931 805 731 | 164 734 417 587 | 527 991 189 421 | 1 496 553 873 125 |
| 24 | 160 996 | 16 777 217 | 375 273 548 | 131 803 805 801 | 525 985 494 172 | 1 780 826 727 893 | 5 301 057 659 396 |
| 25 | 260 497 | 33 554 431 | 864 169 601 | 395 394 640 187 | 1 679 126 417 281 | 6 004 756 243 261 | 18 770 042 517 521 |