NUMBERS COMMON TO TWO POLYGONAL SEQUENCES

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The polygonal sequence (or sequences of polygonal numbers) of order r (where r is an integer, $r \geq 3$) may be defined recursively by

$$(1) \qquad (r,i) = 2(r, i - 1) - (r, i - 2) + r - 2$$

with $(r,0) = 0$, $(r,1) = 1$.

It is possible to obtain a direct formula for (r,i) from (1). A particularly simple way of doing this is via the Gregory interpolation formula. (For an interesting discussion of this formula and its derivation, see [3].) The result is

$$(2) \qquad (r,i) = i + (r - 2)i(i - 1)/2 = [(r - 2)i^2 - (r - 4)i]/2 .$$

It is comforting to note that the "square" numbers — the polygonal numbers of order 4 — actually are the squares of the integers.

Using either (1) or (2), we can take a look at the first few, say, triangular numbers $(r = 3)$

$$0, 1, 3, 6, 10, 15, 21, 28, 36, 45, \cdots .$$

One observation we can make is that three of these numbers are also squares — namely 0, 1, and 36. We can pose the following question: Are there any more of these "triangular-square" numbers? Are there indeed infinitely many of them? What can be said about the numbers common to <u>any</u> pair of polygonal sequences?

We shall begin by answering the last of these questions, and then return to the other two. Suppose that s is an integer common to the polygonal sequences of orders r_1 and r_2 (say $r_1 < r_2$). Then there exist integers p and q such that

$$s = [(r_1 - 2)p^2 - (r_1 - 4)p]/2 = [(r_2 - 2)q^2 - (r_2 - 4)q]/2 ,$$

so that

$$(3) \qquad (r_1 - 2)p^2 - (r_1 - 4)p = (r_2 - 2)q^2 - (r_2 - 4)q ,$$

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and in fact, since both sides of the equation (3) are always even, every pair of non-negative integers p, q which satisfy (3) determine such an integer s.

As a quadratic in p, this has integral solutions, so — since all coefficients are integers — the discriminant

$$(r_1 - 4)^2 + 4(r_1 - 2)(r_2 - 2)q^2 - 4(r_1 - 2)(r_2 - 4)q$$

must be a perfect square, say x^2, so that

$$x^2 = 4(r_1 - 2)(r_2 - 2)q^2 - 4(r_1 - 2)(r_2 - 4)q + (r_1 - 4)^2 .$$

As a quadratic in q, this also has integral solutions, and the discriminant — and hence $1/16^{\text{th}}$ of the discriminant — must again be a perfect squre, say y^2, so that

$$(4) \qquad y^2 - (r_1 - 2)(r_2 - 2)x^2 = (r_1 - 2)^2(r_2 - 4)^2 - (r_1 - 2)(r_2 - 2)(r_1 - 4)^2 ,$$

where p and q are given by

$$(5) \qquad p = \frac{(r_1 - 4) + x}{2(r_1 - 2)} \qquad q = \frac{(r_1 - 2)(r_2 - 4) + y}{2(r_1 - 2)(r_2 - 2)}$$

Although it can be shown, by solving (5) for x and y and substituting into (4), that every solution of (4) gives a solution of (3), it should be noted that some of the integer solutions of (4) may not give integer values for p and q. Nevertheless, (4) and (5) give us all possible candidates for integer solutions of (3).

Now (4) is in the form of Pell's equation, $y^2 - dx^2 = N$, which has a finite number of integral solutions in x and y if d is a perfect square while N does not vanish. For then the left side can be factored into $(y - ax)(y + ax)$, where a is an integer; and N has only finitely many integral divisors.

So we already have a partial answer to our question. If $(r_1 - 2)(r_2 - 2)$ is a perfect square and the quantity on the right side of (4) is non-zero, we have only finitely many candidates for integers common to the two sequences of orders r_1 and r_2.

On the other hand, if $(r_1 - 2)(r_2 - 2)$ is a perfect square and the right side of (4) is zero, then (4) reduces to a linear equation in x and y:

$$y = \pm\sqrt{(r_1 - 2)(r_2 - 2)}\, x \quad .$$

Since the coefficient of x is an integer, this has infinitely many integral solutions.

An analysis of the right side of (4) reveals that, with $r_1 \neq r_2$, this quantity vanishes only when one of r_1 and r_2 is 3 and the other is 6. In that case, (4) becomes $y^2 - 4x^2 = 0$, or $y = \pm 2x$; and equations (5), with y replaced by $\pm 2x$, become $p = (x - 1)/2$; $q = (1 \pm x)/4$.

At this point it is not too hard to see that for infinitely many integers x, the above equations yield non-negative integral values for both p and q. Therefore, there are infinitely many hexagonal-triangular numbers. In this case, however, we have taken the long way around; for it can be shown directly, using (3), that indeed every hexagonal number is also a triangular number.

It remains for us to investigate what happens when $(r_1 - 2)(r_2 - 2)$ is not a perfect square (and here the right side of (4) is necessarily non-zero). If this is the case, then there are infinitely many positive integral solutions to (4) if there is one such solution [2, p. 146], But in fact we can always exhibit at least one solution — namely $x_1 = r_1$, $y_1 = r_2(r_1 - 2)$ — corresponding to $p = q = 1$. We still have the job, however, of showing that infinitely many of these solutions of (4) give us integer solutions of (3).

Consider the related equation

$$u^2 - (r_1 - 2)(r_2 - 2)v^2 = 1 . \tag{6}$$

With $(r_1 - 2)(r_2 - 2)$ not a perfect square, this has infinitely many integral solutions, generated by

$$u_n + v_n \sqrt{(r_1 - 2)(r_2 - 2)} = (u_1 + v_1 \sqrt{(r_1 - 2)(r_2 - 2)})^n ,$$

where u_1, v_1 is the smallest positive solution [2, p. 142]. We obtain u_1, v_1 by inspection. In particular, u_2, v_2, given by

$$u_2 + v_2 \sqrt{(r_1 - 2)(r_2 - 2)} = (u_1 + v_1 \sqrt{(r_1 - 2)(r_2 - 2)})^2 ,$$

is a solution of (6), and by expanding the right side and comparing coefficients, we get

$$\begin{aligned} u_2 &= u_1^2 + (r_1 - 2)(r_2 - 2)v_1^2 \\ v_2 &= 2u_1v_1 \end{aligned} \tag{7}$$

Now infinitely many (but not necessarily all) of the positive solutions of (4) are given by

$$y_{n+1} + x_{n+1}\sqrt{(r_1 - 2)(r_2 - 2)} = (u_i + v_i \sqrt{(r_1 - 2)(r_2 - 2)})(y_n + x_n\sqrt{(r_1 - 2)(r_2 - 2)}) \tag{8}$$

where u_i, v_i is any positive solution of (6) [2, p. 146], say u_2, v_2. Again comparing coefficients, we get

$$\begin{aligned} y_{n+1} &= u_2y_n + (r_1 - 2)(r_2 - 2)v_2x_n , \\ x_{n+1} &= v_2y_n + u_2x_n , \end{aligned} \tag{9}$$

with the side conditions $x_1 = r_1$, $y_1 = r_2(r_1 - 2)$.

Consider the first of equations (9). This can, by adding a suitable quantity to each side, be changed to

$$y_{n+1} + (r_1 - 2)(r_2 - 4) + (r_1 - 2)(r_2 - 4)(u_2 - 1) = u_2(y_n + (r_1 - 2)(r_2 - 4)) + (r_1 - 2)(r_2 - 2)v_2x_n ,$$

and using (6) and (7), we get

$$y_{n+1} + (r_1 - 2)(r_2 - 4) = u_2(y_n + (r_1 - 2)(r_2 - 4)) + 2(r_1 - 2)(r_2 - 2)u_1 v_1 x_n - 2(r_1 - 2)^2(r_2 - 2)(r_2 - 4)v_1^2 . \tag{10}$$

Recalling that $y_1 = r_2(r_1 - 2)$, clearly

$$y_1 \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)(r_2 - 2)} ;$$

and letting $n = k$ in (10), we see that if

$$y_k \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)(r_2 - 2)}$$

for some integer k, then each term on the right of (10) is divisible by $2(r_1 - 2)(r_2 - 2)$. Hence the left side of (10) is divisible by this same quantity, and

$$y_{k+1} \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)(r_2 - 2)} .$$

By mathematical induction, and with reference to the second of equations (5), all of the y_n's given by (9) produce positive integral values for q.

Similarly, the second of equations (9) can be transformed into

$$x_{n+1} + (r_1 - 4) + (u_2 - 1)(r_1 - 4) + v_2(r_1 - 2)(r_2 - 4) = v_2(y_n + (r_1 - 2)(r_2 - 4)) + u_2(x_n + (r_1 - 4)),$$

and again using (6) and (7), we get

$$x_{n+1} + (r_1 - 4) = v_2(y_n + (r_1 - 2)(r_2 - 4)) + u_2(x_n + r_1 - 4)) - 2v_1^2(r_1 - 2)(r_2 - 2)(r_1 - 4) - 2u_1v_1(r_1 - 2)(r_2 - 4) . \tag{11}$$

Since

$$y_n \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)(r_2 - 2)}$$

for all n, certainly

$$y_n \equiv -(r_1 - 2)(r_2 - 4) \pmod{2(r_1 - 2)}\ .$$

We have that

$$x_1 \equiv -(r_1 - 4) \pmod{2(r_1 - 2)}\ ,$$

since $x_1 = r_1$; and it can be seen from (11) that if

$$x_k \equiv -(r_1 - 4) \pmod{2(r_1 - 2)}$$

for some integer k, then

$$x_{k+1} \equiv -(r_1 - 4) \pmod{2(r_1 - 2)}\ .$$

That is, $2(r_1 - 2)$ divides $x_n + (r_1 - 4)$ for every positive integer n.

To summarize, for $(r_1 - 2)(r_2 - 2)$ not a perfect square, we have exhibited (in (9)) infinitely many — but not necessarily all — of the solutions to the Pell-type equation (4); and all of these give positive integral solutions p, q of (3). These, in turn, give integers s which are common to the two polygonal sequences of orders r_1 and r_2.

In view of the above, we can now state the following theorem:

Theorem. Given two distinct integers r_1 and r_2, with $3 \le r_1 < r_2$, each defining the order of a polygonal sequence, there are infinitely many integers common to both sequences if and only if one of the following is true:

i. $r_1 = 3$ and $r_2 = 6$, or

ii. $(r_1 - 2)(r_2 - 2)$ is not a perfect squre.

In practice, given particular integers r_1 and r_2, we can get all of the solutions of (4) by using at most finitely many equations of the form (8), with a different x_1, y_1 for each one. Some of these equations can be eliminated or modified to leave out those solutions which give non-integer values for either p or q. We may then obtain equations generating all pairs p, q for which $(r_1, p) = (r_2, q)$; and, if desired, finitely many equations generating the numbers s common to the two sequences. The procedure for finding all solutions of (4) is arduous and depends erratically on the actual values of r_1 and r_2. For the general machinery, see G. Chrystal [1, pp. 478-486].

Now we can easily answer our questions about triangular squares. Letting $r_1 = 3$ and $r_2 = 4$, $(r_1 - 2)(r_2 - 2)$ becomes 2, which is not a perfect square. There are, then, infinitely many triangular squares. As a matter of fact, this result has been known for some time. To exhibit these numbers, we note that since the coefficient of q in (3) becomes 0, we can get a formula like (4) by applying the quadratic formula only once. The result is

$$x^2 - 8q^2 = 1$$

or

$$x^2 - 2y^2 = 1, \tag{12}$$

where $p = (x - 1)/2$ and $q = y/2$. Conveniently enough, (12) is already in the form of (6); and since $x_1 = 3$, $y_1 = 2$ is the smallest positive solution, all non-negative solutions of (12) are given by

$$x_n + y_n\sqrt{2} = (3 + 2\sqrt{2})^n \qquad (n = 0, 1, 2, \cdots). \tag{13}$$

Certainly the "next" solution is given by

$$x_{n+1} + y_{n+1}\sqrt{2} = (x_n + y_n\sqrt{2})(3 + 2\sqrt{2}),$$

and by comparing coefficients we get

$$\begin{aligned} x_{n+1} &= 3x_n + 4y_n, \\ y_{n+1} &= 2x_n + 3y_n, \end{aligned} \tag{14}$$

with (from (13)) $x_0 = 1$, $y_0 = 0$.

It follows by induction from (14) that all values of y_n are even non-negative integers, and all x_n's are odd positive integers. Therefore, for any solution p, q of (3) — in non-negative integers and with $r_1 = 3$, $r_2 = 4$ — there exists an n $(n = 0, 1, 2, \ldots)$ such that

$$\begin{aligned} p = p_n &= (x_n - 1)/2 \\ q = q_n &= y_n/2 \end{aligned} \tag{15}$$

where x_n, y_n are given by (14). Furthermore, p_n, q_n given by (15) forms a non-negative integral solution for any n, since the x_n's are always odd and all of the y_n's are even.

All triangular square numbers, then, are given by

$$s_n = (p_n^2 + p_n)/2 = q_n^2. \tag{16}$$

Solving (14) with $x_0 = 1$, $y_0 = 0$, we get

$$\begin{aligned} x_n &= [(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n]/2 \\ y_n &= [(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n]/2\sqrt{2}, \end{aligned}$$

and combining these with (15) and (16), we obtain

$$s_n = \frac{(17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2}{32},$$

where s_n is the n^{th} triangular-square number.

Likewise, we can compute a formula for the n^{th} triangular-pentagonal number. The result is

$$s_n = \frac{(2 - \sqrt{3})(97 + 56\sqrt{3})^n + (2 + \sqrt{3})(97 - 56\sqrt{3})^n - 4}{48} .$$

This agrees with a result recently published by W. Sierpiński [4].

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$$M^{-1} = \sum_{k=0}^{\infty} \frac{m_k^*}{k!} D^k$$

is given by

$$\text{(VII)} \qquad \sum_{k=0}^{\infty} \frac{m_k^*}{k!} t^k = 1/(Ae^{c_1 t} + Be^{c_2 t}) .$$

We now note that for Case 2, where $A + B = 0$, Eq. (VII) does not exist for $t = 0$, and hence there is no inverse operator M^{-1}. Thus, a sufficient condition for M^{-1} (see (I)) to exist is that $A + B \neq 0$, i.e., Case 1. For $A + B \neq 0$, one readily finds that

$$\text{(VIII)} \qquad (A + B)m_k^* = (c_2 - c_1)^k H_k\left(\frac{c_1}{c_1 - c_2}\middle| -A/B\right) ,$$

where $H_k(x|\lambda)$ is the Eulerian polynomial cited in (*).

Many more identities can be quoted. Indeed, for $m, n = 0, 1, \cdots$, one has

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