ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-234 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Suppose an alphabet, $A = \{x_1, x_2, x_3, \cdots\}$, is given along with a binary connective, P (in prefix form). Define a well formed formula (wff) as follows: a wff is

(1) x_i for $i = 1, 2, 3, \cdots$, or

(2) If A_1, A_2 are wff's, then PA_1A_2 is a wff and

(3) The only wff's are of the above two types.

A wff of order n is a wff in which the only alphabet symbols are $x_1, x_2, \cdots, x_n$ in that order with each letter occurring exactly once. There is one wff of order 1, namely x_1. There is one wff of order 2, namely Px_1x_2. There are two wff's of order 3, namely $Px_1Px_2x_3$ and $PPx_1x_2x_3$, and there are five wff's of order 4, etc.

Define a sequence
$$\{G_i\}_{i=1}^{\infty}$$
as follows:

g_i is the number of distinct wff's of order i.

a. Find a recurrence relation for $\{G_i\}_{i=1}^{\infty}$ and

b. Find a generating function for $\{G_i\}_{i=1}^{\infty}$.

H-235 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

a. Find the second-order ordinary differential equation whose power series solution is

$$\sum_{n=0}^{\infty} F_{n+1} x^n .$$

b. Find the second-order ordinary differential equation whose power series solution is

$$\sum_{n=0}^{\infty} L_{n+1} x^n .$$

H-236 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$(1) \qquad \sum_{n=0}^{\infty} (-1)^n x^{n^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(x)_{2n}} \prod_{k=1}^{\infty} (1 - x^k) ,$$

$$(2) \qquad \sum_{n=0}^{\infty} (-1)^n x^{(n+1)^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x)_{2n+1}} \prod_{k=1}^{\infty} (1 - x^k) ,$$

where $(x)_k = (1 - x)(1 - x^2) \cdots (1 - x^k)$, $(x)_0 = 1$.

SOLUTIONS

TO COIN A THEOREM

H-199 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

A certain country's coinage consists of an infinite number of types of coins: $\cdots$, C_{-2}, C_{-1}, C_0, C_1, $\cdots$. The value V_n of the coin C_n is related to the others as follows: for <u>all</u> n,

$$V_n = V_{n-3} + V_{n-2} + V_{n-1} .$$

Show that any (finite) pocketful of coins is equal in value to a pocketful containing at most one coin of each type.

Solution by the Proposer.

Call a pocketful Q canonical if it consists entirely of coins of different types and such that no three coins of "adjacent" types (e.g., c_n, c_{n+1} and c_{n+2}) are present. Call two pocketfuls equivalent if they have the same value.

We will prove for any pocketful P the statement:

S: P is equivalent to a canonical pocketful.

Note that any pocketful containing only differing types is equivalent to a canonical pocketful since the three adjacent coins of highest value, C_n, C_{n+1}, C_{n+2} can be replaced by C_{n+3}, etc.

Assume for the moment, the following statement:

R: S is true for any canonical pocketful to which one extra coin has been added.

Then the general result follows by induction on the number of coins for any pocketful P: Remove a coin to get P', apply the induction hypothesis to P' to get a canonical pocketful Q, return the removed coin and apply R.

Now to prove R, let us prove by induction on j the series of statements R_j:

R_j $\left\{\begin{array}{l}\text{If } Q \text{ is any canonical pocketful in which the coin of least value is a } C_n, \text{ then if} \\ \text{a } C_{n+j} \text{ or a } C_{n+j} \text{ and a } C_{n+j+1} \text{ be added to } Q \text{ to get a pocketful } P', \text{ then } S \\ \text{is true for } P'.\end{array}\right.$

Assume R_k for all $k < j$ (it is obvious if $k \le -3$). Now let Q be canonical. We can suppose that $n + j = 0$. Suppose Q contains δ_i coins of type C_i, $\delta_i = 0$ or 1, $\delta_i\delta_{i+1}\delta_{i+2} = 0$ for all i. Then

$$* \qquad Q \cup C_0 \equiv \cdots \delta_{-3},\ \delta_{-2},\ \delta_{-1},\ \delta_0 + 1,\ \delta_1,\ \delta_2,\ \delta_3,\ \cdots .$$

If $\delta_0 = 0$, we are finished, so assume $\delta_0 = 1$. Then

$$Q \cup C_0 \equiv \cdots \delta_{-3} + 1,\ \delta_{-2},\ \delta_{-1},\ 0,\ \delta_1 + 1,\ \delta_2,\ \delta_3,\ \cdots .$$

Again, if $\delta_1 = 0$, by induction, we are finished, so assume $\delta_1 = 1$. Then

$$Q \cup C_0 \equiv \cdots \delta_{-3} + 1,\ \delta_{-2} + 1,\ \delta_{-1},\ 0,\ 0,\ \delta_2 + 1,\ \delta_3 \cdots .$$

Now, since $\delta_0\delta_1\delta_2 = 0$, $\delta_2 = 0$ so we are finished.

For the next part,

$$Q \cup C_0 \cup C_1 \equiv \cdots \delta_{-3},\ \delta_{-2},\ \delta_{-1},\ \delta_0 + 1,\ \delta_1 + 1,\ \delta_2,\ \delta_3 \cdots .$$

If either δ_0 or $\delta_1 = 0$, this case can be handled as above, so suppose δ_0 and δ_1 are 1. Then

$$\begin{array}{rlllllllll}
Q \cup C_0 \cup C_1 \equiv \cdots & \delta_{-3} & \delta_{-2} & \delta_{-1} & 2 & 2 & \delta_2 & \delta_3 & \cdots \\
\equiv \cdots & \delta_{-3}+1 & \delta_{-2}+1 & \delta_{-1}+1 & 1 & 2 & \delta_2 & \delta_3 & \cdots \\
\equiv \cdots & \delta_{-3}+1 & \delta_{-2}+2 & \delta_{-1}+2 & 2 & 1 & \delta_2 & \delta_3 & \cdots \\
\equiv \cdots & \delta_{-3}+1 & \delta_{-2}+2 & \delta_{-1}+1 & 1 & 0 & \delta_2+1 & \delta_3 & \cdots \\
\equiv \cdots & \delta_{-3}+1 & \delta_{-2}+1 & \delta_{-1} & 0 & 1 & \delta_2+1 & \delta_3 & \cdots
\end{array}$$

and again, by induction, we are finished. This completes the proof.

We note, without proof, that no two canonical pocketfuls are equivalent.

Editorial Note: The given sequences identify the elements of the union.

ASYMPTOTIC PI

H-200 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.

Let $M(n)$ be the number of primes (distinct) which divide the binomial coefficient,

$$C_k^n \equiv \binom{n}{k}^* .$$

Clearly, for $1 \le n \le 15$, we have $M(1) = 0$, $M(2) = M(3) = 1$, $M(4) = M(5) = 2$, $M(6) = M(7) = M(8) = M(9) = 3$, $M(10) = 4$, $M(11) = M(12) = M(14) = 5$, $M(13) = M(15) = 6$, etc. Show that

$$\{m(n)\}_{n=1}^{\infty}$$

has an upper bound and find an asymptotic formula for $M(n)$.

*Divide at least one C_k^n, where $0 \le k \le n$.

Solution by D. Singmaster, Instituto Mathematica, Pisa, Italy.

For a prime p, if

$$p \,\Big|\, \binom{n}{k}$$

for some k, $0 \le k \le n$, then $p \mid n!$ and so $p \le n$. Hence $M(n) \le \pi(n)$, where $\pi(n)$ is the number of primes less than or equal to n. We claim $M(n) \sim \pi(n)$. To see this, we can use the following result of B. Ram. (See: L. E. Dickson, History of the Theory of Numbers, Vol. 1; Chelsea, 1952; p. 274, item 98.) There is at most one prime $p < n$ such that

$$p \nmid \binom{n}{k}$$

for $0 \le k \le n$ and such a prime p exists if and only if $n + 1 = ap^s$ with $1 \le a < p < n$.

Since Ram's paper is somewhat inaccessible, I will prove a slight sharpening of it, using an accessible result. N. J. Fine ("Binomial Coefficients modulo a Prime," Amer. Math. Monthly, 54 (1947), 589-592, Theorem 4) has shown that

$$p \nmid \binom{n}{k}$$

for $0 \le k \le n$ if and only if $n = ap^s - 1$ with $1 \le a < p$ and $s \ge 0$. Now suppose we have two primes p_1 and p_2 with $p_1 < p_2 \le n + 1$ and

$$p_i \nmid \binom{n}{k}$$

for $0 \le k \le n$. By Fine's result, we have

$$n + 1 = a_1 p_1^{s_1} = a_2 p_2^{s_2}$$

with $1 \le a_1 < p_1$ and $1 \le a_2 < p_2$. But $a_1 < p_1 < p_2$ implies that $p_2 \nmid a_1 p_1^{s_1}$, so $s_2 = 0$ and $n + 1 = a_2 < p_2$, contrary to $p_2 \le n + 1$. Hence there is at most one prime $p \le n + 1$ such that

$$p \nmid \binom{n}{k}$$

for $0 \le k \le n$ and such a p exists if and only if $n + 1 = ap^s$ with $1 \le a < p$. (More discussion related to this may be found in my survey paper: "Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers," (to appear).)

By carefully examining the role of $n + 1$, we can deduce the following formulas for $M(n)$.

$$M(n) = \begin{cases} \pi(n+1) & \text{if } n+1 \neq ap^s \text{ with } 1 \leq a < p \\ \pi(n+1) - 1 & \text{otherwise}. \end{cases}$$

$$M(n) = \begin{cases} \pi(n) & \text{if } n+1 \neq ap^s \text{ with } 1 \leq a < p \leq n \\ \pi(n) - 1 & \text{otherwise.} \end{cases}$$

Hence $M(n) \sim \pi(n) \sim [(\log n)/n]^{-1}$.

Incidentally, $M(13) = M(15) = 5$, contrary to what was asserted in the statement of the problem. The first place where $M(n) > M(n+1)$ is $n = 83$, where $M(83) = 23$ and $M(84) = 22$. The next cases are $n = 89$ and $n = 104$. From the expression for $M(n)$, we have the following necessary conditions for such an n: $n + 1$ must have three distinct prime factors and $n + 2$ must not be prime.

Also solved by the Proposer.

DISPLAY CASE

H-201 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Copy 1, 1, 3, 8, $\cdots$, F_{2n-2} $(n \geq 1)$ down in staggered columns as in display C:

$$C \qquad \begin{array}{ccccc} 1 & & & & \\ 1 & 1 & & & \\ 3 & 1 & 1 & & \\ 8 & 3 & 1 & 1 & \\ 21 & 8 & 3 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \quad .$$

(i) Show that the row sums are F_{2n+1} $(n = 0, 1, 2, \cdots)$.

(ii) Show that, if the columns are multiplied by 1, 2, 3, $\cdots$, sequentially to the right, then the row sums are F_{2n+2} $(n = 0, 1, 2, \cdots)$.

(iii) Show that the rising diagonal sums $(\nearrow)$ are F^2_{n+1} $(n = 0, 1, 2, \cdots)$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

(i) Let R_n denote the row sum of the $(n+1)^{\text{th}}$ row, $(n = 0, 1, 2, \cdots)$, with $R_0 = 1$.

$$R_n = 1 + \sum_{k=0}^{n-1} F_{2n-2k} = 1 + \sum_{k=1}^{n} F_{2k} = 1 + \sum_{k=1}^{n} (F_{2k+1} - F_{2k-1}) = 1 + F_{2n+1} - 1 = F_{2n+1},$$

as asserted.

(ii) Let S_n denote the sum as defined in the problem, for the $(n+1)^{\text{th}}$ row, $(n = 0, 1, 2, \cdots)$, with $S_0 = 1$. Then, if $n \geq 1$,

$$S_n = \sum_{k=1}^{n} k F_{2n+2-2k} + (n+1) = \sum_{k=0}^{n-1} (n-k) F_{2k+2} + (n+1) = (n+1) + \sum_{k=0}^{n-1} F_{2k+2} \sum_{i=0}^{n-k-1} 1$$

$$= (n+1) + \sum_{i=0}^{n-1} 1 \sum_{k=0}^{n-i-1} F_{2k+2} = (n+1) + \sum_{i=0}^{n-1} (F_{2n-2i+1} - 1) = (n+1) + \sum_{i=1}^{n} F_{2i+1} - n$$

$$= 1 + \sum_{i=1}^{n} (F_{2i+2} - F_{2i}) = 1 + F_{2n+2} - F_2 = F_{2n+2},$$

as asserted. This is also true for $n = 0$.

(iii) Let T_n denote the rising diagonal sums. Then, if $n \geq 2$,

$$T_n = \sum_{k=1}^{\frac{1}{2}n} F_{4k} + 1, \text{ if } n \text{ is even}; \quad T_n = \sum_{k=1}^{\frac{1}{2}(n+1)} F_{4k-2}, \text{ if } n \text{ is odd}; \quad T_0 = T_1 = 1.$$

$$T_{2m} = \sum_{k=1}^{m} F_{4k} + 1 = F_1 + \sum_{k=1}^{m} (F_{4k+1} - F_{4k-1}) = \sum_{i=0}^{2m} (-1)^i F_{2i+1};$$

also,

$$T_{2m+1} = \sum_{k=1}^{m+1} F_{4k-2} = \sum_{k=1}^{m+1} (F_{4k-1} - F_{4k-3}) = \sum_{i=0}^{2m+1} (-1)^{i+1} F_{2i+1}.$$

Combining these results, we have

$$T_n = \sum_{i=0}^{n} (-1)^{n-i} F_{2i+1} = \sum_{i=0}^{n} (-1)^{n-i} (F_{i+1}^2 + F_i^2)$$

$$= \sum_{i=0}^{n} (-1)^{n-i} F_{i+1}^2 - (-1)^{n-i+1} F_i^2 = (-1)^{n-n} F_{n+1}^2 - (-1)^{n+1} \cdot 0 = F_{n+1}^2.$$

This last result is also true for $n = 0$ and $n = 1$.

Also solved by the Proposer and one unsigned solver.