A FIBONACCI ANALOGUE OF GAUSSIAN BINOMIAL COEFFICIENTS

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Gauss, in his work on quadratic reciprocity, defined in [1] an analogue to the binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(x^n - 1)(x^{n-1} - 1) \cdots (x^{n-k+1} - 1)}{(x^k - 1)(x^{k-1} - 1) \cdots (x - 1)} ,$$

n and k positive integers. In order to make the analogy to the binomial coefficients more complete, it is customary to let

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 ,$$

for $n = 0, 1, 2, \cdots$, and

$$\begin{bmatrix} n \\ k \end{bmatrix} = 0$$

for $n < k$. We shall call these rational functions in x, Gaussian binomial coefficients. It is shown in [7] that these functions satisfy the recursion formula:

$$\begin{bmatrix} n \\ k \end{bmatrix} = x^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} ,$$

and if we note that as $x \to 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} \to \binom{n}{k},$$

where

$$\binom{n}{k}$$

is the usual binomial coefficient, then the above recursion formula becomes

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} ,$$

the recursion formula for the binomial coefficients.

Just as the binomial coefficients are always integers, although they appear to be ratios of integers, the Gaussian binomial coefficients are in fact polynomials rather than rational functions. This is easily seen from the recursion formula and mathematical induction. (See [7].) The Gaussian binomial coefficients and their multinomial analogues have some interesting geometric interpretations and combinatorial applications in counting inversions and special partitions of the integers. Some of these appear in [1] and [6].

There is another well known analogue to the binomial coefficients, the so-called "Fibonomial coefficients:"

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1}$$

n, k positive integers, and

$$\binom{n}{0}_F = \binom{n}{n}_F = 1$$

for $n = 0, 1, 2, \cdots$. It is well known that this is always an integer [5].

Let us now examine the Gaussian analogue of the "Fibonomial coefficient:"

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{(x^{F_n} - 1)(x^{F_{n-1}} - 1) \cdots (x^{F_{n-k+1}} - 1)}{(x^{F_k} - 1)(x^{F_{k-1}} - 1) \cdots (x^{F_1} - 1)} ,$$

n, k positive integers and

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_F = \begin{bmatrix} n \\ n \end{bmatrix}_F = 1$$

for $n = 0, 1, 2, \cdots$. Again it is clear that as $x \to 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_F \to \binom{n}{k}_F .$$

Since

$$F_n = F_{k+1} F_{n-k} + F_k F_{n-k-1} ,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{(x^{F_{k+1}F_{n-k} + F_kF_{n-k-1}} - 1)(x^{F_{n-1}} - 1) \cdots (x^{F_{n-k+1}} - 1)}{(x^{F_k} - 1)(x^{F_{k-1}} - 1) \cdots (x^{F_1} - 1)}$$

(1) $$= \frac{(x^{F_{k+1}F_{n-k}+F_kF_{n-k-1}} - x^{F_kF_{n-k-1}} + x^{F_kF_{n-k-1}} - 1)}{(x^{F_k} - 1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F$$

$$= \frac{x^{F_kF_{n-k-1}}(x^{F_{k+1}F_{n-k}} - 1) + (x^{F_kF_{n-k-1}} - 1)}{(x^{F_k} - 1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F$$

$$= x^{F_kF_{n-k-1}}\left(\sum_{i=1}^{F_{k+1}} x^{(F_{k+1}-i)F_{n-k}}\right)\begin{bmatrix} n-1 \\ k \end{bmatrix}_F$$

$$+\left(\sum_{i=1}^{F_{n-k-1}} x^{(F_{n-k-1}-i)F_k}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F$$

so that we have a recursion formula for the "Gaussian Fibonomial coefficients" and this, with mathematical induction, implies the rather remarkable property of these functions: they are polynomials rather than rational functions as they appear to be. Furthermore if we let $x \to 1$ in the recursion formula (1) we obtain

$$\binom{n}{k}_F = F_{k+1}\binom{n-1}{k}_F + F_{n-k-1}\binom{n-1}{k-1}_F$$

the recursion formula for the Fibonomial coefficients. This is the recursion formula used in [3] to prove that the Fibonomial coefficients are integers.

The more general sequence g_n where $g_0 = 0$, $g_1 = 1$, $g_{n+2} = p\cdot g_{n+1} + q\cdot g_n$, $n \geq 0$, $p > 0$, $q \geq 0$, satisfies $g_n = g_{k+1}\cdot g_{n-k} + q\cdot g_k\cdot g_{n-k-1}$ (see [3]) and if we define

$\begin{bmatrix} n \\ k \end{bmatrix}_g$ as follows: $$\begin{bmatrix} n \\ k \end{bmatrix}_g = \frac{(x^{g_n}-1)(x^{g_{n-1}}-1)\cdots(x^{g_{n-k+1}}-1)}{(x^{g_k}-1)(x^{g_{k-1}}-1)\cdots(x^{g_1}-1)}$$

n,k positive integers, and

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_g = \begin{bmatrix} n \\ n \end{bmatrix}_g = 1$$

for $n = 0, 1, 2, \cdots$, then it follows, mutatis mutandis, that

$$\begin{bmatrix} n \\ k \end{bmatrix}_g = x^{q\cdot g_kg_{n-k-1}}\left(\sum_{i=1}^{g_{k+1}} x^{(g_{k+1}-i)\cdot g_{n-k}}\right)\begin{bmatrix} n-1 \\ k \end{bmatrix}_g$$

$$+\left(\sum_{i=1}^{q\cdot g_{n-k+1}} x^{(q\cdot g_{n-k+1}-i)\cdot g_k}\right)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_g .$$

Again,

$$\begin{bmatrix} n \\ k \end{bmatrix}_g$$

are polynomials. Furthermore the functions are again polynomials where $g_n = f_n(t)$, the Fibonacci polynomials, at least for positive integral t, where $f_0(t) = 0$, $f_1(t) = 1$,

$$f_{n+2}(t) = t \cdot f_{n+1}(t) + f_n(t), \qquad n \geq 0 .$$

Since the Pell sequence can be generated as a special case of the Fibonacci polynomials (where $t = 2$), the above "coefficients" are polynomials also when defined in terms of the Pell sequence.

Furthermore, because of the direct analogy between the definitions of the Gaussian binomial coefficients and the related Fibonacci analogues defined above and the expression for the binomial coefficients as ratios of factorials, the polynomials when arranged in a triangular array like Pascal's Triangle will have the beautiful hexagon property described by Hoggatt and Hansell in [4], that the product of the elements "surrounding" an element in the array is a perfect square and the set of six elements can be broken down into two sets of three, the products of the elements in each set being equal. In fact all the perfect square patterns of Usiskin in [8] will appear in these new arrays; the proofs carry over directly.

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