ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

PROBLEMS PROPOSED IN THIS ISSUE

B-280 Proposed by Maxey Brooke, Sweeney, Texas.

Identify A, E, G, H, J, N, O, R, T, V as the ten distinct digits such that the following holds with the dots denoting some seven-digit number and ϕ representing zero:

$$\begin{array}{r} \text{V E R N E R} \\ \times \underline{\qquad\qquad\text{E}} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ -\; \underline{\text{R}\,\phi\,\phi\,\phi\,\phi\,\text{J R}} \\ \text{H O G G A T T} \end{array}$$

B-281 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $T_n = n(n+1)/2$. Find a positive integer b such that for all positive integers m, $T_{11\cdots 1} = 11\cdots 1$, where the subscript on the left side has m 1's as the digits in base b and the right side has m 1's as the digits in base b^2.

B-282 Proposed by Herta T. Freitag, Roanoke, Virginia.

Characterize geometrically the triangles that have

$$L_{n+2}L_{n-1}, \quad 2L_{n+1}L_n, \quad \text{and} \quad 2L_{2n} + L_{2n+1}$$

as the lengths of the three sides.

B-283 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the ordered triple (a, b, c) of positive integers with $a^2 + b^2 = c^2$, a odd, $c < 1000$, and c/a as close to 2 as possible. [This approximates the sides of a 30°, 60°, 90° triangle with a Pythagorean triple.]

B-284 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $z^2 - xy - y = 0$ and let k, m, and n be nonnegative integers. Prove that:

(a) $z^n = p_n(x,y)z + q_n(x,y)$, where p_n and q_n are polynomials in x and y with integer coefficients and p_n has degree $n - 1$ in x for $n > 0$;

(b) There are polynomials r, s, and t, not all identically zero and with integer coefficients, such that

$$z^k r(x,y) + z^m s(x,y) + z^n t(x,y) = 0 .$$

B-285 Proposed by Barry Wolk, University of Manitoba, Winnipeg, Manitoba, Canada.

Show that

$$F_{k(n+1)}/F_k = \sum_{r=0}^{[n/2]} (-1)^{r(k-1)} \binom{n-r}{r} L_k^{n-2r} .$$

SOLUTIONS

A LUCAS PRODUCT

B-256 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $L_{2n} - 3(-1)^n$ is the product of two Lucas numbers.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

$$\begin{aligned} L_{n-1}L_{n+1} &= (\alpha^{n+1} + \beta^{n+1})(\alpha^{n-1} + \beta^{n-1}) \\ &= L_{2n} + (\alpha^2 + \beta^2)(-1)^{n-1} \\ &= L_{2n} - 3(-1)^n . \end{aligned}$$

Also solved by Wray G. Brady, Paul S. Bruckman, James D. Bryant, Tim Carroll, Juliana D. Chan, Warren Cheves, Ralph Garfield, John E. Homer, Graham Lord, F. D. Parker, C. B. A. Peck, M. N. S. Swamy, William E. Thomas, Jr., David Zeitlin, and the Proposer.

A FIBONACCI PRODUCT

B-257 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $[L_{2n} + 3(-1)^n]/5$ is the product of two Fibonacci numbers.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

$$\begin{aligned} F_{n-1} \cdot F_{n+1} &= (\alpha^{n-1} - \beta^{n-1})(\alpha^{n+1} - \beta^{n+1})/5 \\ &= (\alpha^{2n} + \beta^{2n} - (\alpha\beta)^{n-1}(\alpha^2 + \beta^2))/5 \\ &= (L_{2n} + 3(-1)^n)/5 \quad . \end{aligned}$$

Also solved by Wray G. Brady, Paul S. Bruckman, James D. Bryant, Tim Carroll, Juliana D. Chan, Warren Cheves, Ralph Garfield, John E. Homer, F. D. Parker, C. B. A. Peck, M. N. S. Swamy, William E. Thomas, Jr., Gregory Wulczyn, David Zeitlin, and the Proposer.

GOLDEN RATIO FORMULA

B-258 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Let $[x]$ denote the greatest integer in x, $a = (1 + \sqrt{5})/2$, and $e_n = (1 + (-1)^n)/2$. Prove that for all positive integers m and n

(a) $$nF_{n+1} = [naF_n] + e_n$$

(b) $$nF_{m+n} = F_m([naF_n] + e_n) + nF_{m-1}F_n \; .$$

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Since $aF_n = F_{n+1} - b^n$, where $b = (1 - \sqrt{5})/2$, to prove (a) it suffices to show $|nb^n| < 1$. But

$$1 \cdot (\sqrt{5} - 1)/2 < .65 < 1$$

and

$$2 \cdot (\sqrt{5} - 1)^2/4 < 2 \cdot (.65)^2 < 1 \; .$$

The latter inequality verifies the case $n = 2$ of the induction hypothesis: if $n \ge 2$ then $n|b^n| < 1$. Then

$$(n + 1)|b^{n+1}| < (n + 1)(.65)^{n+1} < (n + 1)(.65)/n < 1 \; ,$$

for $n \ge 2$, which completes the induction and the proof of (a).

Equality (b) comes from substituting (a) in the known identity:

$$F_{m+n} = F_m F_{n+1} + F_{m-1}F_n \; .$$

Also solved by C. B. A. Peck and the Proposer.

A. P. OF BINOMIAL COEFFICIENTS

B-259 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Characterize the infinite sequence of ordered pairs of integers (m, r) with $4 \le 2r \le m$, for which the three binomial coefficients

$$\binom{m-2}{r-2}, \quad \binom{m-2}{r-1}, \quad \binom{m-2}{r}$$

are in arithmetic progression.

Solution by Paul Smith, University of Victoria, Victoria, B.C., Canada.

Equivalently, find all solutions of:

$$\binom{m-2}{r} + \binom{m-2}{r-2} = 2\binom{m-2}{r-1}.$$

A simple computation yields $m = (m - 2r)^2$, whence $m = n^2$ and $r = (m - \sqrt{m})/2$; $2r$ is strictly less than m. The required sequence is thus

$$\{(n^2, \ (n^2 - n)/2)\}_{n>2} .$$

Also solved by Wray G. Brady, Paul S. Bruckman, Tim Carroll, Herta T. Freitag, Graham Lord, David Zeitlin, and the Proposer.

SUMS OF DIVISORS

B-260 Proposed by John L. Hunsucker and Jack Nebb, University of Georgia, Athens, Georgia.

Let $\sigma(n)$ denote the sum of the positive integral divisors of n. Show that

$$\sigma(mn) > \sigma(m) + \sigma(n)$$

for all integers $m > 1$ and $n > 1$.

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

We may write

$$m = \prod_{k=1}^{r} p_k^{e_k}, \qquad n = \prod_{k=1}^{r} p_k^{f_k}, \qquad mn = \prod_{k=1}^{r} p_k^{e_k+f_k},$$

where the p_k are distinct primes and the e_k and f_k are nonnegative integers. Since

$$\sigma(m) = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{e_k}),$$

one has

$$\sigma(m)/m = \prod_{k=1}^{r} \sum_{j=0}^{e_k} p^{-j} .$$

Then it follows that $\sigma(mn)/mn > \sigma(m)/m$ and $\sigma(mn)/mn > \sigma(n)/n$. We may add these inequalities and multiply by mn, which yields:

$$2\sigma(mn) > n\sigma(m) + m\sigma(n) \geq 2\sigma(m) + 2\sigma(n)$$

and the desired result follows.

Also solved by Wray G. Brady, Tim Carroll, Graham Lord, C. B. A. Peck, Philip Tracy, and the Proposer.

CYCLIC GROUP MODULO D

B-261 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let d be a positive integer and let S be the set of all non-negative integers n such that $2^n - 1$ is an integral multiple of d. Show that either $S = \{0\}$ or the integers in S form an infinite arithmetic progression.

Solution by Tim Carroll, Western Michigan University, Kalamazoo, Michigan.

$0 \in S$ since $d \mid (2^0 - 1)$. Let n be the least positive integer in S when $S \neq \{0\}$. For any positive integer k,

$$2^{kn} - 1 = (2^n - 1)(2^{n(k-1)} + 2^{n(k-2)} + \cdots + 2^n + 1) .$$

Since d divides $2^n - 1$, d divides $2^{kn} - 1$ for all positive k. Therefore $kn \in S$ for all positive integers k. We now show there are no other integers in S. Suppose $m \in S$ and $m = qn + r$, $0 < r < n$.

$$\begin{aligned} 2^m - 1 &= 2^{qn}2^r - 1 \\ &= 2^{qn}2^r - 2^{qn} + 2^{qn} - 1 \\ &= 2^{qn}(2^r - 1) + (2^{qn} - 1) . \end{aligned}$$

Since $q_n \in S$, $m \in S$, and d does not divide 2^{qn}, d divides $2^r - 1$. But this is impossible by our choice of n. Therefore, $S = \{0\}$ or $S = \{0, n, 2n, 3n, \cdots\}$.

Also solved by Wray G. Brady, Paul S. Bruckman, Warren Cheves, Herta. T. Freitag, Graham Lord, Richard W. Sielaff, Paul Smith, David Zeitlin, and the Proposer.