COMBINATIONS AND SUMS OF POWERS

MYRON TEPPER
195 Dogwood, Park Forest, Illinois 60466

We adopt the following notation and conventions:

1. n and Q are non-negative integers.

2. $$S_Q = \sum_{i=1}^{n} i^Q .$$

3. $$\sum_{i=a}^{b} F(i) = 0 \quad \text{for} \quad a > b .$$

4. $$\prod_{i=a}^{b} F(i) = 1 \quad \text{for} \quad a > b .$$

5. $B_1 = 1/6$, $B_2 = -1/30$, $B_3 = 1/42$, etc., are the non-zero Bernoulli numbers.

6. $$g_Q(x_1, x_2, \cdots, x_m) = \left[\prod_{i=1}^{m} x_i^{-1}\right] \cdot \left[\prod_{j=1}^{m-1} \binom{x_{j+1}}{x_{j-1}}\right] \cdot \binom{Q+1}{x_m - 1} .$$

For example,

$$g_4(1) = 1^{-1}\binom{5}{0}$$

$$g_4(1,3) = (1\cdot 3)^{-1}\binom{3}{0}\binom{5}{2}$$

$$g_4(1,3,4) = (1\cdot 3\cdot 4)^{-1}\binom{3}{0}\binom{4}{2}\binom{5}{3} .$$

7. $$d_Q(x_1, x_2, \cdots, x_m) = g_Q(x_1, x_2, \cdots, x_m)\cdot n^{x_1} .$$

<u>Theorem 1.</u> Say $Q \geq 0$. Then

$$(Q+1)S_Q = n^{Q+1} + (Q+1)n^Q - 1 + \prod_{i=1}^{Q} (1 - r_i) ,$$

where

$$\prod_{i=2}^{Q} (1 - r_i)$$

is expressed in terms of sums of products of the r_i, and for each such product, e.g., $r_{x_1} \cdot r_{x_2} \cdot \cdots \cdot r_{x_m}$, where $x_1 < x_2 < \cdots < x_m$ for $m \geq 2$, we let $r_{x_1} \cdot r_{x_2} \cdot \cdots \cdot r_{x_m} = dQ(x_1, x_2, \cdots, x_m)$.

Theorem 2. Say $Q \geq 1$. Then

$$(2Q + 1)B_Q = -r_1 \prod_{i=2}^{2Q} (1 - r_i) ,$$

where

$$-r_1 \prod_{i=2}^{2Q} (1 - r_i)$$

is expressed in terms of sums of products of the r_i, and for each such product, e.g., $r_{x_1} \cdot r_{x_2} \cdot \cdots \cdot r_{x_m}$, where $x_1 < x_2 < \cdots < x_m$ for $m \geq 2$, we let $r_{x_1} \cdot r_{x_2} \cdot \cdots \cdot r_{x_m} = g_{2Q}(x_1, x_2, \cdots, x_m)$.

Theorem 3. Say $Q \geq 1$. Then

$$(S + 1)^Q - S^Q = (n + 1)^Q - 1 ,$$

where S^i is formally replaced by S_i when the left-hand side of this equation is expanded; e.g., $1S_0 + 3S_1 + 3S_2 = (n + 1)^3 - 1$. Hence, starting with $S_0 = n$, this theorem can be used to find S_Q in a recursive fashion.

Theorem 4.

$$S_1 = \frac{1}{2!} \begin{vmatrix} 1 & n \\ 1 & n^2 \end{vmatrix} + n$$

$$S_2 = \frac{1}{3!} \begin{vmatrix} 1 & 0 & n \\ 1 & 2 & n^2 \\ 1 & 3 & n^3 \end{vmatrix} + n^2$$

$$S_3 = \frac{1}{4!} \begin{vmatrix} 1 & 0 & 0 & n \\ 1 & 2 & 0 & n^2 \\ 1 & 3 & 3 & n^3 \\ 1 & 4 & 6 & n^4 \end{vmatrix} + n^3 ,$$

etc., where the entries in the determinants are binomial coefficients, zeros, and powers of n.

We now illustrate two more methods for finding S_Q.

Method 1. The "$(i + 1)^Q - (i - 1)^Q$" method. For example,

$$\sum_{i=1}^{n} \left[(i + 1)^2 - (i - 1)^2 \right] = \sum_{i=1}^{n} 4i \; .$$

$$\therefore \qquad (n + 1)^2 + n^2 - 1 = \sum_{i=1}^{n} 4i \; .$$

$$\therefore \qquad 4 \sum_{i=1}^{n} i = 2n^2 + 2n \; .$$

$$\therefore \qquad \sum_{i=1}^{n} i = \frac{n^2 + n}{2} = \frac{n(n + 1)}{2} \; .$$

Method 2. Lagrange interpolation. Assuming that S_Q is a polynomial of degree $Q + 1$ in n, we now compute S_1. Let $f(n) = S_1 = 1 + 2 + \cdots + n$. Then, by Lagrange interpolation, we have $f(n) = f(1)P_1 + f(2)P_2 + f(3)P_3$, where, letting $t_i = i$,

$$P_1 = \frac{(n - t_2)(n - t_3)}{(t_1 - t_2)(t_1 - t_3)} = \frac{(n - 2)(n - 3)}{(-1)(-2)}$$

$$P_2 = \frac{(n - t_1)(n - t_3)}{(t_2 - t_1)(t_2 - t_3)} = \frac{(n - 1)(n - 3)}{(1)(-1)}$$

$$P_3 = \frac{(n - t_1)(n - t_2)}{(t_3 - t_1)(t_3 - t_2)} = \frac{(n - 1)(n - 2)}{(2)(1)} \; .$$

Editors' Note: This abstract qualifies for the Fibonacci Note Service. It is an abstract of a paper which is fifty pages long. If you would like a Xerox copy of the entire article at four cents a page (which includes postage, materials and labor), send your request to:

Brother Alfred Brousseau, St. Mary's College, Moraga, California 94575.