LATIN k-CUBES

JOSEPH ARKIN
197 Old Nyack Turnpike, Spring Valley, New York 10977
and
E. G. STRAUS
University of California, Los Angeles 90024

1. LATIN SQUARES

A Latin square of order n is an $n \times n$ square in which each of the numbers $0, 1, \cdots, n-1$ occurs exactly once in each row and exactly once in each column. For example

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{matrix}$$

are Latin squares of order $2, 3, 4,$ respectively. Two Latin squares of order n are orthogonal, if when one is superimposed on the other, every ordered pair $00, 01, \cdots, n-1\ n-1$ occurs. Thus

$$\begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{matrix} \quad \text{superimpose to} \quad \begin{matrix} 00 & 11 & 22 \\ 12 & 20 & 01 \\ 21 & 02 & 10 \end{matrix}$$

and therefore are orthogonal squares of order 3. A set of Latin squares of order n is orthogonal if every two of them are orthogonal. As an example the 4×4 square of triples

$$\begin{matrix} 000 & 111 & 222 & 333 \\ 123 & 032 & 301 & 210 \\ 231 & 320 & 013 & 102 \\ 312 & 203 & 130 & 021 \end{matrix}$$

represents three mutually orthogonal squares of order 4 since each of the 16 pairs $00, 01, \cdots, 33$ occurs in each of the three possible positions among the 16 triples.

There cannot exist more than $n-1$ mutually orthogonal Latin squares of order n, and the existence of such a complete system is equivalent to the existence of a finite projective plane of order n, that is a system of n^2+n+1 points and n^2+n+1 lines with $n+1$ points on each line. If n is a power of a prime there exist finite fields of order n which can be used to construct finite projective planes of order n. So, for $n = 2, 3, 4, 5, 7, 8, 9$ there exist complete systems of $n-1$ orthogonal Latin squares of order n. We have listed the examples $n = 2, 3, 4,$ above. It is known [2] that there are no orthogonal Latin squares of order 6 and that there are at least two orthogonal Latin squares of every order $n > 2$, $n \neq 6$. In fact, the number of mutually orthogonal Latin squares of order n goes to infinity with n [3]. However no case of a complete system of $n-1$ orthogonal Latin squares is known for any n which is not a power of a prime.

2. LATIN CUBES

We can generalize all these concepts to $n \times n \times n$ cubes and cubes of higher dimensions.

A Latin cube of order n is an $n \times n \times n$ cube (n rows, n columns and n files) in which the numbers $0, 1, \cdots, n-1$ are entered so that each number occurs exactly once in each row, column and file. If we list the cube in terms of the n squares of order n which form its different levels we can list the cubes

$$\begin{array}{ccc} 0\ 1 & & 1\ 0 \\ 1\ 0 & & 0\ 1 \end{array} \quad \text{and} \quad \begin{array}{ccccc} 0\ 1\ 2 & & 1\ 2\ 0 & & 2\ 0\ 1 \\ 1\ 2\ 0 & & 2\ 0\ 1 & & 0\ 1\ 2 \\ 2\ 0\ 1 & & 0\ 1\ 2 & & 1\ 2\ 0 \end{array}$$

as Latin cubes of order two and three, respectively. Since even this method of listing becomes unwieldy for higher dimensions we also use a listing by indices. Thus we write the first cube as $A = (a_{ijk})$ with $a_{000} = 1, a_{010} = 1, a_{011} = 0, a_{100} = 1, a_{101} = 0, a_{110} = 0, a_{111} = 1.$ In a similar manner we can describe four-dimensional cubes $A = (a_{ijk\ell})$ or order $n,$ where each of the indices, i,j,k,ℓ ranges from 1 to $n.$ Generally we can discuss k-cubes $A = (a_{i_1 i_2 \cdots i_k})$ with k indices ranging from 1 to $n.$ These cubes will be Latin k-cobes of order n if each of the n^k entries $a_{i_1 \cdots i_k}$ is one of the numbers $0, 1, \cdots, n-1$ so that $a_{i_1 \cdots i_k}$ ranges over all these numbers as one of the indices varies from 1 to n while the other indices remain fixed.

Orthogonality of Latin cubes is now a relation among three cubes, or in general among k Latin k-cubes. That is, three Latin cubes of order n are orthogonal if, when superimposed, each ordered triple will occur. Similarly k Latin k-cubes are orthogonal if, when superimposed, each ordered k-tuple will occur. A set of at least k Latin k-cubes is orthogonal if every k of its cubes are orthogonal.

Theorem. If there exist two orthogonal Latin squares of order n then there exist 4 orthogonal Latin cubes of order n and k orthogonal Latin k-cubes for each $k > 3.$

Proof. Let $A = (a_{ij}),\ B = (b_{ij})$ be orthogonal Latin squares of order $n.$

Define 4 cubes C, D, E, F of order n by

$$c_{ijk} = a_{a_{ij},k},\quad d_{ijk} = a_{b_{ij},k},\quad c_{ijk} = b_{a_{ij},k},\quad f_{ijk} = b_{b_{ij},k},\quad i,j,k = 0, 1, \cdots, n-1.$$

Note that the squares A,B are used both as entries and as indices in the construction of the cubes. For example the pair of 3×3 Latin squares

$$\begin{array}{ccccc} 0\ 0 & & 1\ 1 & & 2\ 2 \\ 1\ 2 & & 2\ 0 & & 0\ 1 \\ 2\ 1 & & 0\ 2 & & 1\ 0 \end{array}$$

leads to the four $3 \times 3 \times 3$ cubes

$$\begin{array}{lccccc} & 0\ 1\ 2 & & 1\ 2\ 0 & & 2\ 0\ 1 \\ C: & 1\ 2\ 0 & & 2\ 0\ 1 & & 0\ 1\ 2 \\ & 2\ 0\ 1 & & 0\ 1\ 2 & & 1\ 2\ 0 \\[4pt] & 0\ 1\ 2 & & 1\ 2\ 0 & & 2\ 0\ 1 \\ D: & 2\ 0\ 1 & & 0\ 1\ 2 & & 1\ 2\ 0 \\ & 1\ 2\ 0 & & 2\ 0\ 1 & & 0\ 1\ 2 \\[4pt] & 0\ 2\ 1 & & 1\ 0\ 2 & & 2\ 1\ 0 \\ E: & 2\ 1\ 0 & & 0\ 2\ 1 & & 1\ 0\ 2 \\ & 1\ 0\ 2 & & 2\ 1\ 0 & & 0\ 2\ 1 \\[4pt] & 0\ 2\ 1 & & 1\ 0\ 2 & & 2\ 1\ 0 \\ F: & 1\ 0\ 2 & & 2\ 1\ 0 & & 0\ 2\ 1 \\ & 2\ 1\ 0 & & 0\ 2\ 1 & & 1\ 0\ 2 \end{array}$$

Superimposed these lead to a cube of quadruples

$$\begin{array}{lccccc} \text{CDEF:} & 0000\ 1122\ 2211 & & 1111\ 2200\ 0022 & & 2222\ 0011\ 1100 \\ & 1221\ 2010\ 0102 & & 2002\ 0121\ 1210 & & 0110\ 1202\ 2021 \\ & 2112\ 0201\ 1020 & & 0220\ 1012\ 2101 & & 1001\ 2120\ 0212 \end{array}$$

where each ordered triple occurs in every one of the four possible positions in the quadruples.

It is easy to see that C,D,E,F are Latin cubes. For example, for fixed i,j the values $c_{ijk} = a_{a_{ij},k}$ go through the $a_{ij}{}^{th}$ row of $A,$ that is, through the values $0, 1, \cdots, n-1.$ For fixed i,k the index a_{ij} goes through all the values in the i^{th} row of $A,$ that is, through all values $0, 1, \cdots, n-1$ and hence c_{ijk} goes through all values in the k^{th} column of $A.$ Finally for fixed j,k the index a_{ij} goes through all values in the j^{th} column of A and therefore c_{ijk} again goes through all values in the k^{th} column of $A.$

To prove the orthogonality of, say, C,D,E we have to prove that for every triple (x,y,z) from $\{0, 1, \cdots, n-1\}$ the equations

$$c_{ijk} = x, \qquad d_{ijk} = y, \qquad e_{ijk} = z$$

have a solution i,j,k. By the orthogonality of A and B the pair (x,z) occurs exactly once in the superimposed square AB so that the equations $a_{\ell,k}=x$, $b_{\ell,k}=z$ determine k and ℓ. Thus the equations

$$c_{ijk} = a_{a_{ij},k} = x \qquad \text{and} \qquad e_{ijk} = b_{a_{ij},k} = z$$

determine a_{ij} and k. Now, since A is a Latin square, there is exactly one occurrence of y in the k^{th} column of A so the equation

$$d_{ijk} = a_{b_{ij},k} = y$$

determines b_{ij} and the pair (a_{ij},b_{ij}) determines i,j;

Thus for every triple (x,y,z) there is a unique triple (i,j,k).

This construction is essentially that given by Arkin for 4 orthogonal $10 \times 10 \times 10$ cubes [1].

To prove the last part of the theorem we proceed by induction on k. Let $A^1, \cdots, A^k$ be orthogonal Latin k-cubes of order n, and write the entries of A^j as $a^j_{i_1,\cdots,i_k}$. We now define $k+1$ orthogonal Latin $(k+1)$-cubes $B^1, \cdots, B^{k+1}$ by

$$\begin{aligned} b^1_{i_1,\cdots,i_{k+1}} &= a_{a^1_{i_1,\cdots,i_k},i_{k+1}} \\ &\vdots \\ b^k_{i_1,\cdots,i_{k+1}} &= a_{a^k_{i_1,\cdots,i_k},i_{k+1}} \\ b^{k+1}_{i_1,\cdots,i_{k+1}} &= b_{a^1_{i_1,\cdots,i_k},i_{k+1}} \; . \end{aligned}$$

We omit the proof that the B^j are Latin cubes, which is the same as before. In order to prove orthogonality we have to solve

$$B^j_{i_1\cdots i_{k+1}} = x_j \qquad j = 1, \cdots, k+1 \; .$$

For any $(k+1)$-tuple $(x_1, \cdots, x_{k+1})$ from $\{0,1, \cdots, n-1\}$. Now, by the orthogonality of A and B the two equations

$$A_{a^1_{i_1,\cdots,i_k},i_{k+1}} = x_1, \qquad B_{a^1_{i_1,\cdots,i_{k+1}},i_{k+1}} = x_{k+1}$$

determine $a^1_{i_1\cdots i_k}$ and i_{k+1}. Once i_{k+1} is determined the equations

$$A_{a^j_{i_1,\cdots,i_k},i_{k+1}} = x_j \qquad j = 2, \cdots, k$$

determine

$$a^j_{i_1\ldots i_k} \qquad (j = 2, \cdots, k).$$

Once the elements

$$a^j_{i_1\cdots i_k} \qquad (j = 1, \cdots, k)$$

are determined it follows from the orthogonality of the k-cubes $A^1, \cdots, A^k$ that the indices $i_1, \cdots, i_k$ are determined. Thus for every $(k+1)$-tuple $(x_1, \cdots, x_{k+1})$ there is a unique $(k+1)$-tuple $(i_1, \cdots, i_{k+1})$ with

$$B^j_{i_1\cdots i_{k+1}} = x_j \qquad j = 1, \cdots, k+1 .$$

Since, as we mentioned, there are orthogonal Latin squares of every order $n > 2$, $n \neq 6$ we have the following.

Corollary. There exist orthogonal k-tuples of Latin k-cubes of order n for every $n > 2$, $n \neq 6$.

3. FINITE FIELDS

A field is a system of elements closed under the rational operations of addition, subtraction, multiplication and division (except by 0) subject to the usual commutative, associative and distributive laws. There exist finite fields with n elements if and only if n is a power of a prime p. The prime p is the characteristic of the field and we have $p\alpha = 0$ for every α in the field. Following are the addition and multiplication tables for the fields with 3 and 4 elements:

F_3

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

F_4

$+$	0	1	α	$1+\alpha$
0	0	1	α	$1+\alpha$
1	1	0	$1+\alpha$	α
α	α	$1+\alpha$	0	1
$1+\alpha$	$1+\alpha$	α	1	0

$\times$	0	1	α	$1+\alpha$
0	0	0	0	0
1	0	1	α	$1+\alpha$
α	0	α	$1+\alpha$	1
$1+\alpha$	0	$1+\alpha$	1	α

.

If there is a field F_n with n elements, that is if n is a power of a prime, we use the elements $\{f_1, f_2, \cdots, f_n\}$ of F_n as indices to construct Latin squares, cubes, etc. We give the construction for cubes, but the generalization to k-cubes is easily seen.

Let α, β, γ be three nonzero elements of F_n then we can define the Latin cube $A = (a_{ijk})$ by

$$a_{ijk} = \alpha f_i + \beta f_j + \beta f_k \ .$$

To see that A is a Latin cube consider, say, fixed i,j and see that γf_k runs through all elements of F_n as f_k does. Hence a_{ijk} runs through F_n as $k = 1, \cdots, n$.

Now let (α,β,γ), (α',β',γ') and $(\alpha'',\beta'',\gamma'')$ be three triples of nonzero elements of F_n so that the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \neq 0.$$

Then the three Latin cubes

$$A = (a_{ijk}), \quad A' = (a'_{ijk}), \quad A'' = (a''_{ijk})$$

with

$$a_{ijk} = \alpha f_1 + \beta f_j + \gamma f_k, \quad a'_{ijk} = \alpha' f_i + \beta' f_j + \gamma' f_k, \quad a''_{ijk} = \alpha'' f_i + \beta'' f_j + \gamma'' f_k$$

are orthogonal. This follows from the fact that for any triple (x,y,z) from F_n the three equations

$$a_{ijk} = x, \quad a'_{ijk} = y, \quad a''_{ijk} = z$$

have a unique solution f_i, f_j, f_k.

Now the Vandermonde determinants

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)$$

are different from zero for any three distinct elements α,β,γ of F_n. Thus, letting α run through the nonzero elements of F_n we get $n-1$ orthogonal Latin cubes of order n,

$$A^\alpha = (a^\alpha_{ijk}), \quad a^\alpha_{ijk} = f_i + \alpha f_j + \alpha^2 f_k \ .$$

The construction for a system of $n-1$ orthogonal Latin k-cubes of order n proceeds in exactly the same way if we set

$$A^\alpha = (a^\alpha_{i_1\cdots i_k}), \quad a^\alpha_{i_1\cdots i_k} = f_{i_1} + \alpha f_{i_2} + \cdots + \alpha^{k-1} f_{i_k}$$

where α runs through the nonzero elements of F_n.

Theorem. If n is a power of a prime and $k < n$, then there exists a system of $n-1$ orthogonal k-cubes of order n.

Our previous examples constructing four orthogonal cubes of orders 3 and 4 show that $n-1$ is not necessarily the maximal number of orthogonal k-cubes of order n for $k > 2$. However, the orthogonal cubes constructed with the aid of finite fields satisfy additional properties. For each fixed value of k the squares

$$A^\alpha_{\bullet\bullet k} = (a^\alpha_{ijk}) \qquad i,j = 1, 2, \cdots, n$$

form a complete system of $n-1$ orthogonal Latin squares as α ranges through the nonzero elements of F_n, and similarly for each fixed i the squares

$$A^\alpha_{i\bullet\bullet} = (a^\alpha_{ijk}) \qquad j,k = 1, 2, \cdots, n$$

form a complete system of orthogonal Latin squares. If n is a power of 2 then the third family of cross-sections

$$A^\alpha_{\bullet j\bullet} = (a^\alpha_{ijk}) \qquad i,k = 1, 2, \cdots, n$$

form a complete system of orthogonal Latin squares for each fixed j, while for n odd we get a system of $(n-1)/2$ orthogonal Latin squares, each square occurring twice.

Theorem. If n is a power of 2 then there exist $n-1$ orthogonal Latin cubes of order n with the property that the corresponding plane sections form systems of $n-1$ orthogonal Latin squares.

If n is a power of an odd prime then there exist $n-1$ orthogonal Latin cubes with the property that the corresponding plane cross-sections in two directions form complete systems of orthogonal Latin squares, while the plane cross-sections in the third direction form a system of $(n-1)/2$ orthogonal Latin squares, each square occurring twice.

Finally we observe that if we have orthogonal k-cubes of orders m and n then we can form their Kronecker products to obtain orthogonal k-cubes of order mn. That is from orthogonal k-cubes

$$A^1 = (a^1_{i_1 \cdots i_k}), \cdots, A^\ell = (a^\ell_{i_1 \cdots i_k}); \qquad B^1 = (b^1_{i_1 \cdots i_k}), \cdots, \qquad B^\ell = (b^\ell_{i_1 \cdots i_k}),$$

where the a's run from 1 to m and the b's from 1 to n we can form the orthogonal k-cubes $C^1, \cdots, C^\ell$, where

$$C^j = (c^j_{i_1 \cdots i_k}) \qquad \text{and} \qquad c^j_{i_1 \cdots i_k} = (a^j_{i_1 \cdots i_k}, b^j_{j_1 \cdots j_k})$$

so that the c's run through all ordered pairs $(1,1), \cdots, (m,n)$ as the pairs $(i_1, j_1), \cdots, (i_k, j_k)$ run through these ordered pairs. Thus we have the following.

Corollary. If

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \qquad \text{and} \qquad q = \min_{1 \le j \le s} p_j^{\alpha_j}$$

then for any $k < q$ there exist at least $q - 1$ orthogonal Latin k-cubes of order n.

The relation to finite k-dimensional projective spaces is not as immediate as it is for Latin squares, and we shall not discuss it here.

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ON EXTENDING THE FIBONACCI NUMBERS TO THE NEGATIVE INTEGERS

M. G. MONZINGO
Southern Methodist University, Dallas, Texas 75275

A sequence of positive integers defined by the formula

(1) $$x_{n+1} = ax_n + bx_{n-1}, \quad n \text{ a positive integer,}$$

is said to be extendable to the negative integers if (1) holds for n any integer. See page 28 of [1]. The purpose of this note is to show that the Fibonacci numbers form a sequence which is extendable to the negative integers in a unique way. In this note only nontrivial integral sequences will be considered.

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