A PENTAGONAL ARCH

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A pentagonal arch can be generated by rolling a regular pentagon along a baseline as follows. In Fig. 1, as the left-hand pentagon is rolled toward the right, the vertex A moves first to B, then to C, D and finally to E as the successive sides touch the baseline. Connecting these points by line segments, the five-sided polygonal arch $ABCDE$ is formed.

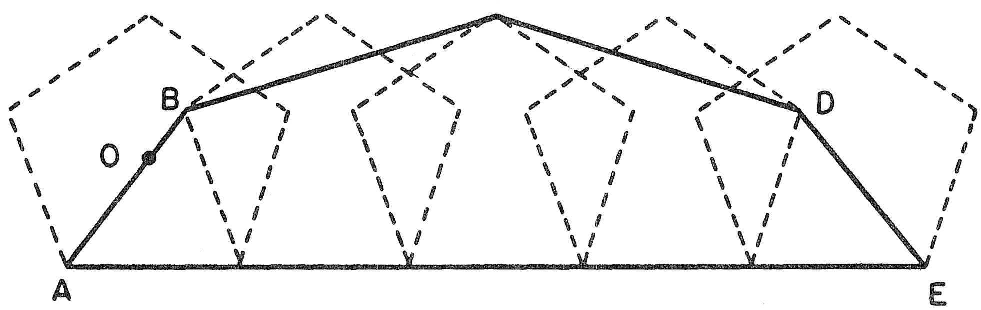

Figure 1

Let s denote the sides of the generating pentagon, and let $\tau = \frac{1}{2}(\sqrt{5}+1)$ denote the golden ratio. It is then easy to show

$$AB = DE = \sqrt{3-\tau}\,s, \qquad BC = CD = \sqrt{2+\tau}\,s$$

$$\angle EAB = \angle AED = 54^\circ, \qquad \angle ABC = \angle BCD = \angle CDE = 144^\circ.$$

Thus the pentagonal arch has some unexpected properties:

(1) Sides AB and BC (and of course DE and CD) are in the proportion of the golden ratio: $\frac{BC}{AB} = \tau$;

(2) The center O of the generating pentagon (in its initial position) lies on the line passing through A and B;

(3) The obtuse angles of the arch are equal.

While these three properties follow directly from the above formulas, a fourth property requires some additional considerations.

(4) $$\frac{\text{area of arch}}{\text{area of generating pentagon}} = 3.$$

To see this first observe from Fig. 1 that it is enough to show that region $ABCFG$ of Fig. 2a is equal in area to that of the generating pentagon.

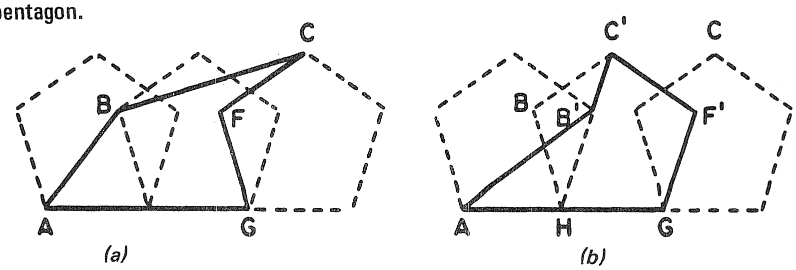

Figure 2

But by referring to Fig. 2b it is seen

$$\text{area } ABCFG = \text{area } AB'C'F'GH = \text{area } HBC'F'G$$

and so property (4) is demonstrated.

In the way of generalization it is natural to ask: Are there analogous properties for the n-sided arch generated by rolling a regular n-gon? The answer is that, upon replacing "pentagon" by "regular polygon," properties (2), (3) and (4) apply equally well to the general case. The two acute base angles are each

$$\left(\frac{1}{2}-\frac{1}{n}\right) \times 180^\circ$$

and the $n-2$ obtuse angles are each equal to

$$\left(1-\frac{1}{n}\right) \times 180^\circ .$$

A proof of (4) for the general case is the main content of [1]; as might be expected the above proof for the pentagonal arch does not generalize, though the ideas are useful for the simpler cases $n = 3, 4, 6$.

There is one aspect of the pentagonal arches which does seem more interesting than for the general arch. By property (2) five arches can be fit together in such a way that their bases form a regular pentagon.

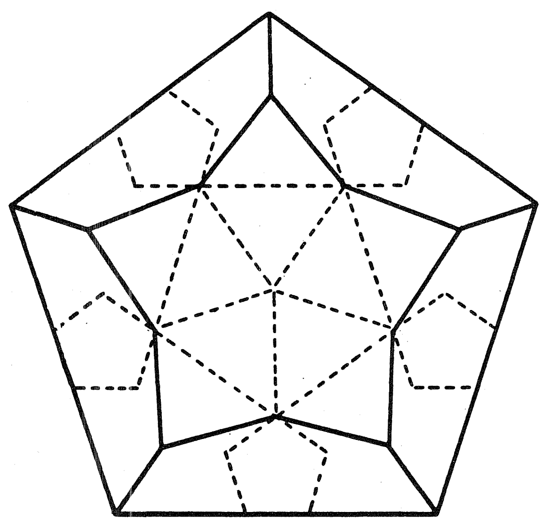

Figure 3

The interior star region can then be partitioned into ten congruent isoceles triangles, each of which has area equal to that of the original generating pentagon. Hence all of the twenty-five elemental polygons of Fig. 3 have equal area.

REFERENCE

1. D.W. DeTemple, "The Area of a Polygonal Arch Generated by Rolling a Polygon," *Amer. Math. Monthly,* (to appear).

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