A METHOD OF CARLITZ APPLIED TO THE K^{TH} POWER GENERATING FUNCTION FOR FIBONACCI NUMBERS

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1. INTRODUCTION

If we consider $f(x)$ such that the power series expansion of $f(x)$ is given by

(1.1) $$f(x) = \sum_{n=0}^{\infty} f_n x^n \quad ,$$

then $f(x)$ is called the ordinary generating function of the sequence $\{ f_n \}$.

We define the generating function for the k^{th} power of f_n as

(1.2) $$f_k(x) = \sum_{n=0}^{\infty} f_n^k x^n \quad .$$

The complexity of expressions which involve f_n^k increases as k increases. This makes it increasingly difficult to determine $f_k(x)$ by the methods described by Hoggatt and Lind [2]. Riordan [5] devised a method to overcome this. His approach depended basically on the expansion of f_n^k by the binomial theorem and subsequent examination of the coefficients. Carlitz [1] applied this to the more general relation

(1.3) $$u_n = pu_{n-1} + qu_{n-2} \quad (n > 2), \qquad u_0 = 1, \qquad u_1 = p \ .$$

He then developed an elegant approach which employed a special function of x and z and depended for success on the identity $u_{n+1}u_{n-1} - u_n^2 = q^n$. Because it is so elegant and because it has appeared hitherto in abbreviated form in papers by Carlitz, Riordan, and Horadam [3], it is proposed here to apply it to the Fibonacci sequence and to expound it in sufficient detail for the general reader to be able to follow it. It is worth pointing out that Kolodner [4] used another approach in which he exploited the fact that the zeros of $z^2 - 2z\cos\theta + 1$, with any θ real or complex, are $e^{i\theta}$ and $e^{-i\theta}$, the powers of which are easily managed.

2. CARLITZ' METHOD

Following Carlitz, we write

(2.1) $$F(x,z) = \sum_{k=1}^{\infty} (1 - a^k x)(1 - b^k x) f_k(x) \frac{z^k}{k} \quad ,$$

where $a = \tfrac{1}{2}(1 + \sqrt{5})$ and $b = \tfrac{1}{2}(1 - \sqrt{5})$ satisfy the auxiliary equation $x^2 - x - 1 = 0$. If we expand this, $F(x,z)$ using the power series expansion of $\log(1 + z)$, we find that

$$F(x,z) = \sum_{k=1}^{\infty}(1-a^k+b^k)x+(ab)^kx^2)\frac{z^k}{k}\sum_{j=0}^{\infty}f_j^kx^j$$

$$= -\sum_{j=0}^{\infty}x^j\log(1-f_jz)+\sum_{j=0}^{\infty}x^{j+1}\log(1-af_jz)$$

$$+\sum_{j=0}^{\infty}x^{j+1}\log(1-bf_jz)-\sum_{j=0}^{\infty}x^{j+2}\log(1+f_jz)$$

$$= -\log(1-f_0z)+x\log(1+f_{-1}z)$$

$$+x\sum_{j=0}^{\infty}x^j\log(1-(a+b)f_jz+abf_j^2z^2)$$

$$-x\sum_{j=0}^{\infty}x^j\log(1-f_{j+1}z)-x\sum_{j=0}^{\infty}x^j\log(1+f_{j-1}z).$$

Since $f_{j+1}f_{j-1}-f_j^2=(-1)^{j-1}$, it follows that

$$\begin{aligned}(1-f_{j+1}z)(1+f_{j-1}z) &= 1-(f_{j+1}-f_{j-1})z-f_{j+1}f_{j-1}z^2\\ &= 1-f_jz-(f_j^2-(-1)^j)z^2\ .\end{aligned}$$

These last two lines are the crucial steps because they make it possible to eliminate terms in z from the numerator in the next few lines. It is the inability to do this with higher degree equations that seems to make the method break down then as will be pointed out later.

$$F(x,z) = -\log(1-f_0z)+x\log(1+f_{-1}z)$$

(2.2)
$$+x\sum_{j=0}^{\infty}x^j\log(1-f_jz-f_j^2z^2)$$

$$-x\sum_{j=0}^{\infty}x^j\log(1-f_jz-(f_j^2-(-1)^j)z^2).$$

The last two terms can be combined to give

$$x\sum_{j=0}^{\infty}x^j\left\{-\log\left[1+\frac{(-1)^jz^2}{1-f_jz-f_j^2z^2}\right]\right\},$$

where there is no z in the numerator. This becomes

$$x\sum_{j=0}^{\infty}x^j\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\ \frac{(-1)^{rj}z^{2r}}{(1-f_jz-f_j^2z^2)^r}$$

(2.3)
$$= x\sum_{j=0}^{\infty}x^j\sum_{r=1}^{\infty}\frac{(-1)^{r+rj}}{r}z^{2r}\sum_{k=2r}^{\infty}a_{kr}(f_jz)^{k-2r}\ .$$

The numbers a_{kr} are, in a sense, the "r^{th} convoluted Fibonacci numbers;" they are generated by the r^{th} power of the ordinary generating function for Fibonacci members. They will be considered in more detail in Section 4. (2.3) becomes

$$x \sum_{j=0}^{\infty} x^j \sum_{r=1}^{\infty} \frac{(-1)^{r+rj}}{r} \sum_{k=2r}^{\infty} a_{kr} f_j^{k-2r} z^k$$

$$= x \sum_{j=0}^{\infty} x^j \sum_{k=1}^{\infty} z^k \sum_{r=1}^{[k/2]} \frac{(-1)^{r+rj}}{r} a_{kr} f_j^{k-2r}$$

in which $[k/2]$ is the greatest integer function: it represents the integral part of the real number $k/2$.

If we replace this in (2.2) we get

$$F(x,z) = -\log(1 - f_0 z) + x \log(1 + f_{-1} z)$$

$$+ x \sum_{k=1}^{\infty} z^k \sum_{r=1}^{[k/2]} \frac{(-1)^r}{r} a_{kr} \sum_{j=0}^{\infty} f_j^{k-2r} ((-1)^r x)^j \tag{2.4}$$

$$= -\log(1 - f_0 z) + x \log(1 + f_{-1} z) + x \sum_{k=1}^{\infty} z^k \sum_{r=1}^{[k/2]} \frac{(-1)^r}{r} a_{kr} f_{k-2r}((-1)^r x) .$$

Comparing coefficients of z^k we get

$$\frac{1}{k}(1 - \ell_k x + (-1)^k x^2) f_k(x) = \frac{f_0^k}{k} - x\frac{(-f_{-1})^k}{k} + x \sum_{r=1}^{[k/2]} \frac{(-1)^r}{r} a_{kr} f_{k-2r}((-1)^r x) ,$$

where ℓ_k is the k^{th} Lucas number. Thus,

$$(1 - \ell_k x + (-1)^k x^2) f_k(x) = 1 + kx \sum_{r=1}^{[k/2]} (-1)^r (a_{kr}/r) f_{k-2r}((-1)^r x) , \tag{2.5}$$

which agrees with the result obtained by Riordan's method [5]. For example, put $k=2$, and

$$(1 - 3x + x^2) f_2(x) = 1 + 2x(-1)(1) f_0(-x) = 1 - \frac{2x}{1+x}$$

which gives

$$f_2(x) = \frac{1-x}{1 - 2x - 2x^2 + x^3} .$$

3. THE COEFFICIENTS OF $f_k(x)$

It is still necessary to look more closely at the coefficients, especially for high k. Carlitz' approach here is also rewarding to study. Applying his method to the Fibonacci coefficients we get from before

$$f_k(x) = \sum_{n=0}^{\infty} \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}} \right)^k x^n$$

$$= 5^{k/2} \sum_{s=0}^{k} \binom{k}{s} \left\{ a^{k-s} b^s + a^{2k-2s} b^{2s} x + a^{3k-3s} b^{3s} x^2 + \cdots \right\} \tag{3.1}$$

$$= 5^{k/2} \sum_{s=0}^{k} \binom{k}{s} a^{k-s} b^s (1 - a^{k-s} b^s x)^{-1} .$$

Define,

$$D_k(x) = \prod_{s=0}^{k} (1 - a^{k-s}b^s x)$$

and write $f_k(x) = F_k(x)/D_k(x)$, where $F_k(x)$ is a polynomial of degree $< k\ (k \geq 1)$. We show that the coefficients of these polynomials satisfy certain recurrence relations and can be determined explicitly.

$$f_{k+1}(x) = \sum_{n=0}^{\infty} \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right)^k \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right) x^n$$

(3.2)
$$= \sum_{n=0}^{\infty} \frac{a}{\sqrt{5}} \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right)^k (ax)^n - \frac{b}{\sqrt{5}} \left(\frac{a^{n+1} - b^{n+1}}{\sqrt{5}}\right)^k (bx)^n$$

$$= \frac{a}{\sqrt{5}} f_k(ax) - \frac{b}{\sqrt{5}} f_k(bx)\ .$$

Then

(3.3)
$$\frac{F_{k+1}(x)}{D_{k+1}(x)} = \frac{a}{\sqrt{5}} \frac{F_k(ax)}{D_k(ax)} - \frac{b}{\sqrt{5}} \cdot \frac{F_k(bx)}{D_k(bx)}\ .$$

Now,

$$\frac{D_{k+1}(x)}{D_k(ax)} = \frac{\prod_{s=0}^{k+1} (1 - a^{k+1-s}b^s x)}{\prod_{s=0}^{k} (1 - a^{k+1-s}b^s x)} = (1 - b^{k+1}x)\ .$$

Similarly,

$$\frac{D_{k+1}(x)}{D_k(bx)} = (1 - a^{k+1}x)$$

Whence from (3.3) we get

(3.4)
$$F_{k+1}(x) = \frac{a}{\sqrt{5}} (1 - b^{k+1}x)F_k(ax) - \frac{b}{\sqrt{5}} (1 - a^{k+1}x)F_k\ bx)\ .$$

Put

(3.5)
$$F_k(x) = \sum_{s=0}^{k} F_{ks}x^s$$

and it follows from (3.4) if we equate coefficients of x^j that

(3.6)
$$\begin{aligned} F_{k+1,j} &= \frac{a^{j+1}}{\sqrt{5}} F_{kj} - \frac{a^j b^{k+1}}{\sqrt{5}} F_{k,j-1} - \frac{b^{j+1}}{\sqrt{5}} F_{kj} + \frac{a^{k+1}b^j}{\sqrt{5}} F_{k,j-1} \\ &= f_j F_{kj} + (-1)^k f_{-(k-j+2)} F_{k,j-1} \end{aligned}$$

which is an expression that enables us to find $F_k(x)$ explicitly. We still need to find D_k and to do this we need the following piece of algebra.

It can be shown easily that

$$\prod_{s=0}^{3}(1-z^s x) = (-1)^0 z^0 x^0 + (-1)^1 \frac{(z^4-1)}{(z-1)} z^0 x^1$$

(3.7)
$$+(-1)^2 \frac{(z^4-1)(z^3-1)}{(z-1)(z^2-1)} z^1 x^2 + (-1)^3 \frac{(z^4-1)(z^3-1)(z^2-1)}{(z-1)(z^2-1)(z^3-1)} z^3 x^3$$

$$+(-1)^4 \frac{(z^4-1)(z^3-1)(z^2-1)(z-1)}{(z-1)(z^2-1)(z^3-1)(z^4-1)} z^6 x^4 = \sum_{s=0}^{4} (-1) z^{\frac{1}{2}s(s-1)} \begin{bmatrix} 4 \\ s \end{bmatrix} x^s ,$$

where

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} 4 \\ s \end{bmatrix} = \frac{(z^4-1)(z^3-1)\cdots(z^{4-s+1}-1)}{(z-1)(z^2-1)\cdots(z^s-1)} \quad (s > 0) .$$

More generally we have that

(3.8)
$$\prod_{s=0}^{k}(1-z^s x) = \sum_{s=0}^{k+1} (-1)^s z^{\frac{1}{2}s(s-1)} \begin{bmatrix} k+1 \\ s \end{bmatrix} x^s ,$$

where

$$\begin{bmatrix} k+1 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} k+1 \\ s \end{bmatrix} = \frac{(z^{k+1}-1)(z^k-1)\cdots(z^{k-s+2}-1)}{(z-1)(z^2-1)\cdots(z^s-1)} \quad (s > 0) .$$

In

$$\begin{bmatrix} k+1 \\ s \end{bmatrix}$$

replace z by b/a, and

$$\begin{bmatrix} k+1 \\ s \end{bmatrix} = \frac{((b/a)^{k+1}-1)((b/a)^k-1)\cdots((b/a)^{k-s+2}-1)}{((b/a)-1)((b/a)^2-1)\cdots((b/a)^s-1)}$$

$$= \frac{a^{\frac{s}{2}(1+s)}(b^{k+1}-a^{k+1})(b^k-a^k)\cdots(b^{k-s+2})}{a^{\frac{s}{2}(2k-s+3)}(b-a)(b^2-a^2)\cdots(b^s-a^s)}$$

$$= a^{-ks+s(s-1)} \frac{f_k f_{k-1}\cdots f_{k-s+1}}{f_0 f_1 \cdots f_{s-1}} = a^{-ks+s(s-1)} \left\{ \begin{matrix} k \\ s \end{matrix} \right\} .$$

Thus if we replace x by $a^k x$ in (3.8) we get

(3.9)
$$D_k(x) = \sum_{s=0}^{k+1} (-1)^{\frac{1}{2}s(s+1)} \left\{ \begin{matrix} k \\ s \end{matrix} \right\} x^s ,$$

since $ab = -1$. This completes the examination of the nature of the coefficients of $f_k(x)$.

4. CONVOLUTED FIBONACCI NUMBERS

We shall now review briefly the so-called "convoluted" Fibonacci numbers [5]. a_{kj} satisfies the recurrence relation

(4.1)
$$a_{kj} - a_{k-1,j} - a_{k-2,j} = a_{k-2,j-1}, \quad k > 2j+2 .$$

Moreover, it is convenient to write

$$a_{kj} = 0, \quad k < 2j .$$

By definition,

$$a_j(x) = \sum_{k=2j}^{\infty} a_{kj} x^{k-2j} .$$

Consider

$$\begin{aligned}(1-x-x^2(a_j(x) &= a_{2j,j} + (a_{2j+1,j} - a_{2j,j})x + (a_{2j+2,j} - a_{2j+1,j} - a_{2j,j})x^2 + \cdots \\ &= a_{2j,j} + a_{2j-1,j}x + a_{2j-1,j-1}x + a_{2j,j-1}x^2 + \cdots \\ &= a_{2j,j} + a_{2j-1,j}x - a_{2j-2,j-1} + a_{2j-2,j-1} + a_{2j-1,j-1}x + a_{2j,j-1}x^2 + \cdots \\ &= a_{j-1}(x)\end{aligned}$$

since $a_{kj} = 0,\ k < 2j$. Thus

(4.2) $$(1-x-x^2)^j a_j(x) = (1-x-x^2)^{j-1} a_{j-1}(x) = (1-x-x^2)^{j-2} a_{j-2}(x) = (1-x-x^2)a_1(x) = 1\,.$$

Hence

(4.3) $$a_j(x) = (1-x-x^2)^{-j} = \left\{f(x)\right\}^j\,,$$

where $f(x)$ is the ordinary generating function for Fibonacci numbers.

5. PROBLEMS FOR FURTHER STUDY

Consider the third-order recurrence relation.

(5.1) $$K_n = K_{n-1} + K_{n-2} + K_{n-3} \quad (n > 3)$$

and the sequences

$$\begin{aligned}&0,\ 1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ \cdots,\ K_n,\ \cdots \\ &1,\ 0,\ 1,\ 2,\ 3,\ 6,\ 11,\ 20,\ 37,\ \cdots,\ L_n,\ \cdots\end{aligned}$$

in which

$$L_0 = K_1 - K_0, \quad L_1 = K_2 - K_1\,,$$

and for $n > 2$,

$$L_n = K_{n-1} + K_{n-2}\,.$$

Using a simple induction proof and matrix and determinant theory, we can show that

(5.2) $$\begin{vmatrix} K_{n+1} & K_{n-1} & K_n \\ K_n & K_{n-2} & K_{n-1} \\ K_{n-1} & K_{n-3} & K_{n-2}\end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{vmatrix}^n = 1\,.$$

Similar treatment with a fourth-order recurrence relation and the sequences

$$\begin{aligned}&0,\ 0,\ 1,\ 1,\ 2,\ 4,\ 8,\ 15,\ 29,\ 56,\ \cdots.\ M_n,\ \cdots \\ &0,\ 1,\ 0,\ 1,\ 2,\ 4,\ 7,\ 14,\ 27,\ 52,\ \cdots,\ N_n,\ \cdots \\ &1,\ 0,\ 0,\ 1,\ 2,\ 3,\ 6,\ 12,\ 23,\ 43,\ \cdots,\ O_n,\ \cdots\end{aligned}$$

yields

(5.3) $$\begin{vmatrix} M_{n+3} & M_{n+2} & M_{n+1} & M_n \\ M_{n+2} & M_{n+1} & M_n & M_{n-1} \\ M_{n+1} & M_n & M_{n-1} & M_{n-2} \\ M_n & M_{n-1} & M_{n-2} & M_{n-3}\end{vmatrix} = (-1)^n$$

Ordinary generating functions for these are easily found, but what about generating functions for the powers of the numbers? The forms of (5.2) and (5.3) by comparison with

$$u_{n+1}u_{n-1} - u_n^2 = q^2 \quad \text{and} \quad f_{n+1}f_{n-1} - f_n^2 = (-1)^{n-1}$$

rule out Carlitz' method for finding the k^{th} power generating function for third- and fourth-order recurrence relations. The complexity of the multinomial coefficients would seem to make Riordan's approach break down. Kolodner's dependence on quadratic equation theory makes it difficult to extend his method to general cubic and quartic equations. What approaches then can be used for recurrence relations of order greater than the second?

REFERENCES

1. L. Carlitz, "Generating Functions for Powers of a Certain Sequence of Numbers," *Duke Math. J.,* 29 (1962), pp. 521–537.
2. V.E. Hoggatt, Jr., and D. A. Lind, "A Primer for the Fibonacci Numbers: Part VI," *The Fibonacci Quarterly,* Vol. 5, No. 4 (Dec. 1967), pp. 445-460.

3. A.F. Horadam, "Generating Functions for Powers of a Certain Generalized Sequence of Numbers," *Duke Math. J.,* 32 (1965), pp. 437–446.
4. J.J. Kolodner, "On a Generating Function Associated with Generalized Fibonacci Sequences," *The Fibonacci Quarterly,* Vol. 3, No. 3 (Oct. 1965), pp. 272–278.
5. J. Riordan, "Generating Functions for Powers of Fibonacci Numbers," *Duke Math. J.,* 29(1962), pp. 5–12.

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A CONSTRUCTED SOLUTION OF $\sigma(n) = \sigma(n+1)$

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With $\sigma(n)$ the sum of the positive divisors of n, one finds that

(1) $$\sigma(n) = \sigma(n+1)$$

for

(2) $$n = 14, 206, \cdots, 18873, 19358, \cdots, 174717, \cdots .$$

Sierpinski [1] asked if (1) has infinitely many solutions. Earlier, Erdös had conjectured [2] that it does, but the answer is unknown. Makowski [3] listed the nine solutions of (1) with $n < 10^4$ and subsequently Hunsucker *et al* continued and found 113 solutions with $n < 10^7$. See [4] for a reference to this larger table.

It is unlikely that there are only finitely many solutions but, in any case, there is a much larger solution, namely,

(3) $$n = 5559060136088313 .$$

It is easily verified that the first, second, and fourth examples in (2) are given by

(4) $$n = 2p, \quad n+1 = 3^m q ,$$

where

(4a) $$q = 3^{m+1} - 4, \qquad p = (3^m q - 1)/2$$

are both prime, and m equals $1, 2,$ or 4. One finds that

(4b) $$\sigma(n) = \sigma(n+1) = \frac{1}{2}(9^{m+1} + 3) - 6 \cdot 3^m .$$

The third and fifth examples in (2) are given by

(5) $$n = 3^m q, \qquad n+1 = 2p$$

with the primes

(5a) $$q = 3^{m+1} - 10, \qquad p = (3^m q + 1)/2$$

for $m = 4$ and 5. Then

(5b) $$\sigma(n) = \sigma(n+1) = \frac{1}{2}(9^{m+1} + 9) - 15 \cdot 3^m .$$

Our new solution (3) is given by $(5 - 5a)$ for $m = 16$. But there are no other examples of (5) or (4) for $m < 44$. While we do conjecture that there are infinitely many solutions of (1) we do not think that infinitely many solutions can be constructed in this way. D.H. and Emma Lehmer assisted us in these calculations.

REFERENCES

1. Waclaw Sierpiński, *Elementary Theory of Numbers,* Polska Akademia Nauk, Warsaw, Poland, 1964, p. 166.
2. Paul Erdös, "Some Remarks on Euler's ϕ Function and Some Related Problems," *Bull. AMS,* Vol. 51, 1945, pp. 540–544.
3. A. Makowski, "On Some Equations Involving Functions $\phi(n)$ and $\sigma(n)$," *Amer. Math. Monthly,* Vol. 67, 1960, pp. 668–670; "Correction," *ibid.,* Vol. 68, 1961, p. 650.
4. M. Lal, C. Eldridge, and P. Gillard, "Solutions of $\sigma(n) = \sigma(n+k)$," UMT 37, *Math. Comp.,* Vol. 27, 1973, p. 676.

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