FIBONACCI NOTES
3: q-FIBONACCI NUMBERS

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1. It is well known (see for example [2, p. 14] and [1]) that the number of sequences of zeros and ones of length n:

$$(a_1, a_2, \cdots, a_n) \qquad (a_i = 0 \text{ or } 1) \tag{1.1}$$

in which consecutive ones are forbidden is equal to the Fibonacci number F_{n+2}. Moreover if we also forbid $a_1 = a_n = 1$, then the number of allowable sequences is equal to the Lucas number L_{n-1}. More precisely, for the first problem, the number of allowable sequences with exactly k ones is equal to the binomial coefficient

$$\binom{n-k+1}{k} \quad ;$$

for the second problem, the number of sequences with k ones is equal to

$$\binom{n-k+1}{k} - \binom{n-k-1}{k-2} \quad .$$

We now define the following functions. Let

$$f(n,k) = \sum q^{a_1+2a_2+\cdots+na_n} \quad , \tag{1.2}$$

where the summation is extended over all sequences (1.1) with exactly k ones in which consecutive ones are not allowable. Also define

$$g(n,k) = {\sum}' q^{a_1+2a_2+\cdots+na_n} \quad , \tag{1.3}$$

where the summation is the same as in (1.2) except that $a_1 = a_n = 1$ is also forbidden. We shall show that

$$f(n,k) = q^{k^2} \begin{bmatrix} n-k+1 \\ k \end{bmatrix} \tag{1.4}$$

and

$$g(n,k) = q^{k^2} \begin{bmatrix} n-k+1 \\ k \end{bmatrix} - q^{n+(k-1)^2} \begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} \quad , \tag{1.5}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)} \quad , \tag{1.6}$$

the q-binomial coefficient.

These results suggest that we define q-Fibonacci and q-Lucas numbers by means of

$$F_{n+1}(q) = \sum_{2k \leqslant n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} \quad , \tag{1.7}$$

$$L_n(q) = F_{n+2}(q) - q^n F'_{n-2}(q) \, , \tag{1.8}$$

where

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$$(1.9) \qquad F'_{n+1}(q) = \sum_{2k \leqslant n} q^{(k+1)^2} \left[\begin{matrix} n-k \\ k \end{matrix} \right] .$$

It follows from the definitions that

$$(1.10) \qquad \begin{cases} F_{n+1}(q) - F_n(q) = q^{n-1} F_{n-1}(q) \\ F'_{n+1}(q) - F'_n(q) = q^{n+1} F'_{n-1}(q) . \end{cases}$$

Thus

$$(1.11) \qquad L_n(q) = F_{n+1}(q) + q^n (F_n(q) - F'_{n-2}(q)) .$$

However, $L_n(q)$ does not seem to satisfy any simple recurrence.

2. For the first problem as defined above it is convenient to define $f_j(n,k)$ as the number of allowable sequences with exactly k ones and $a_n = j$, where $j = 0$ or 1. It then follows at once that

$$(2.1) \qquad f_0(n,k) = f_0(n-1, k) + f_1(n-1, k) \qquad (n > 1)$$

and

$$(2.2) \qquad f_1(n,k) = q^n f_0(n-1, k-1) \qquad (n > 1) .$$

Also it is clear from the definition that

$$(2.3) \qquad f(n,k) = f_0(n,k) + f_1(n,k) .$$

Hence, by (2.1),

$$(2.4) \qquad f(n,k) = f_0(n+1, k) .$$

Combining (2.1) and (2.2) we get

$$(2.5) \qquad f_0(n,k) = f_0(n-1, k) + q^{n-1} f_0(n-2, k-1) \qquad (n > 2) .$$

This formula evidently holds for $k = 0$ if we define $f(n, -1) = 0$.

It is convenient to put

$$(2.6) \qquad f_0(0,k) = \begin{cases} 1 & (k=0) \\ 0 & (k>1) . \end{cases}$$

Also, from the definition,

$$(2.7) \qquad f_0(1,k) = \begin{cases} 1 & (k=0 \\ 0 & (k>1) \end{cases}$$

and

$$(2.8) \qquad f_0(2,k) = \begin{cases} 1 & (k=0) \\ q & (k=1) \\ 0 & (k>1) \end{cases} .$$

It follows that (2.5) holds for $n \geqslant 2$.

Now put

$$(2.9) \qquad \Phi(x,y) = \sum_{n,k=0}^{\infty} f_0(n,k) x^n y^k .$$

Then, by (2.6), (2.7) and (2.5),

$$\Phi(x,y) = 1 + x + \sum_{n=2}^{\infty} \sum_{k} \left\{ f_0(n-1, k) + q^{n-1} f(n-2, k-1) \right\} x^n y^k$$

$$= 1 + x\Phi(x,y) + qx^2 y \Phi(qx, y) ,$$

so that

$$(2.10) \qquad \Phi(x,y) = \frac{1}{1-x} + \frac{qx^2 y}{1-x} \Phi(qx, y) .$$

Iteration of (2.10) leads to the series

(2.11)
$$\Phi(x,y) = \sum_{k=0}^{\infty} \frac{q^{k^2}x^{2k}y^k}{(x)_{k+1}} ,$$

where
$$(x)_{k+1} = (1-x)(1-qx)\cdots(1-q^kx).$$

Since
$$\frac{1}{(x)_{k+1}} = \sum_{s=0}^{\infty} \begin{bmatrix} k+s \\ k \end{bmatrix} x^s ,$$

where
$$\begin{bmatrix} k+s \\ s \end{bmatrix}$$

is defined by (1.6), it follows that

$$\Phi(x,y) = \sum_{k=0}^{\infty} q^{k^2}x^{2k}y^k \sum_{s=0}^{\infty} \begin{bmatrix} k+s \\ s \end{bmatrix} x^s$$

$$= \sum_{n=0}^{\infty} \sum_{2k \leqslant n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} x^n y^k .$$

Comparison with (2.9) gives

(2.12)
$$f_0(n,k) = q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} .$$

Therefore, by (2.4),

(2.13)
$$f(n,k) = q^{k^2} \begin{bmatrix} n-k+1 \\ k \end{bmatrix} .$$

3. If we put

(3.1)
$$f(n) = \sum_{2k \leqslant n+1} f(n,k) ,$$

it is evident that
$$f(n) = \Sigma q^{a_1+2a_2+\cdots+na_n} ,$$

where the summation is over all zero-one sequences of length n with consecutive ones forbidden. This suggests that we define

(3.2)
$$F_{n+1}(q) = f(n-1) = \sum_{2k \leqslant n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} \qquad (n \geqslant 0) .$$

We may also define

(3.3)
$$F_0(q) = 0, \qquad F_1(q) = 1 .$$

The next few values are

$$F_2(q) = 1, \qquad F_3(q) = 1+q$$
$$F_4(q) = 1+q+q^2, \qquad F_5(q) = 1+q+q^2+q^3+q^4$$
$$F_6(q) = 1+q+q^2+q^3+2q^4+q^5+q^6$$
$$F_7(q) = 1+q+q^2+q^3+2q^4+2q^5+2q^6+q^7+q^8+q^9 .$$

It is evident from the above that $F_n(1) = F_n$, the ordinary Fibonacci number. To get a recurrence for $F_n(q)$ we use

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} .$$

Then, by (3.2),

$$F_{n+1}(q) - F_n(q) = \sum_k q^{k^2}\left(\begin{bmatrix} n-k \\ k \end{bmatrix} - \begin{bmatrix} n-k-1 \\ k \end{bmatrix}\right) = \sum_k q^{k^2} \cdot q^{n-2k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}$$

$$= q^{n-1}\sum_k q^{(k-1)^2}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} = q^{n-1}\sum_k q^{k^2}\begin{bmatrix} n-k-2 \\ k \end{bmatrix},$$

so that

(3.4) $$F_{n+1}(q) = F_n(q) + q^{n-1}F_{n-1}(q) \qquad (n \geq 1).$$

This of course reduces to the familiar recurrence $F_{n+1} = F_n + F_{n-1}$ when $q = 1$.

It follows easily from (3.4) that $F_n(q)$ is a polynomial in q with positive integral coefficients. If $d(k)$ denotes the degree of $F_k(q)$ then $d(1) = d(2) = 0$, $d(3) = 1$, $d(4) = 2$, $d(5) = 4, \cdots$. Generally it is clear from (3.4) that

(3.5) $$d(n+1) = n - 1 + d(n-1) \qquad (n > 1).$$

Thus

$$d(2n+1) = 2n - 1 + d(2n-1), \qquad d(2n) = 2n - 2 + d(2n-2),$$

which yields

(3.6) $$d(2n+1) = n^2, \qquad d(2n) = n(n-1).$$

If we replace q by q^{-1} we find that

$$\begin{bmatrix} n \\ k \end{bmatrix} \to q^{k^2-nk}\begin{bmatrix} n \\ k \end{bmatrix}.$$

Hence

(3.7) $$F_{n+1}(q^{-1}) = \sum_{2k \leq n} q^{k^2-nk}\begin{bmatrix} n-k \\ k \end{bmatrix}.$$

It follows that

(3.8) $$\begin{cases} q^{n^2}F_{2n+1}(q^{-1}) = \displaystyle\sum_{k=0}^{n} q^{(n-k)^2}\begin{bmatrix} 2n-k \\ k \end{bmatrix} \\ q^{n(n-1)}F_{2n}(q^{-1}) = \displaystyle\sum_{k=0}^{n-1} q^{(n-k)(n-k-1)}\begin{bmatrix} 2n-k-1 \\ k \end{bmatrix} \end{cases}.$$

It follows from (2.11) and (3.2) that

(3.9) $$\sum_{n=0}^{\infty} F_{n+1}(q)x^n = \sum_{k=0}^{\infty} \frac{q^{k^2}x^{2k}}{(x)_{k+1}}.$$

G.E. Andrews proposed the following problem. Show that $F_{p+1}(q)$ is divisible by $1 + q + \cdots + q^{p-1}$, where p is any prime $\equiv \pm 2 \pmod 5$. For proof see [3]. This result is by no means apparent from (3.2). The proof depends upon the identity

(3.10) $$F_{n+1} = \sum_{k=-r}^{r} (-1)^k x^{\frac{1}{2}k(5k-1)}\begin{bmatrix} n \\ e(k) \end{bmatrix},$$

where

$$e(k) = [\tfrac{1}{2}(n+5k)], \qquad r = [\tfrac{1}{5}(n+2)]$$

In general it does not seem possible to simplify the right member of (3.9). However when $x = q$ it is noted in [3] that

(3.11) $$1 + \sum_{n=1}^{\infty} F_n(q)q^n = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k} = \prod_{n=0}^{\infty} (1 - x^{5n+1})^{-1}(1 - x^{5n+4})^{-1}.$$

4. We now turn to the second problem described in the Introduction. To determine $g(n,k)$ as defined in (1.3) it is clear that

(4.1) $$g(n,k) = f(n,k) - h(n,k),$$

where $h(n,k)$ denotes the number of zero-one sequences $(a_1,a_2,\cdots,a_n)$ with k ones, consecutive ones forbidden and in addition $a_1=a_n=1$. Then $a_2=a_{n-1}=0$ while a_3 and a_{n-2} (if they occur) are arbitrary. Thus, for $n\geqslant 4$, $k\geqslant 2$,

$$h(n,k) = q^{n+1+2(k-2)}f(n-4,k-2) = q^{n+2k-3}f(n-2,k-2),$$

so that (4.1) becomes

$$g(n,k) = f(n,k) - q^{n+2k-3}f(n-4,k-2). \tag{4.2}$$

Combining with (2.13) we get

$$g(n,k) = q^{k^2}\begin{bmatrix} n-k+1 \\ k \end{bmatrix} - q^{n+(k-1)^2}\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} \qquad (n\geqslant 4,\ k\geqslant 2). \tag{4.3}$$

As for the excluded values, it is clear that

$$g(n,0) = 1, \qquad g(n,1) = q\begin{bmatrix} n \\ 1 \end{bmatrix} \qquad (n\geqslant 1). \tag{4.4}$$

Also it is easily verified that

$$g(3,k) = 0 \qquad (k\geqslant 2),$$

so that (4.3) holds for all $n\geqslant 1$. It is convenient to define

$$g(0,0) = 1, \qquad g(0,k) = 0 \qquad (k>0). \tag{4.5}$$

Now put

$$g(n) = \sum_{2k\leqslant n+1} g(n,k). \tag{4.6}$$

Then by (3.2) and (4.3) we have

$$g(n) = f(n) - q^n f'(n-4), \tag{4.7}$$

where

$$f'(n) = \sum_{2k\leqslant n+1} q^{(k+2)^2}\begin{bmatrix} n-k+1 \\ k \end{bmatrix}. \tag{4.8}$$

It is easily verified that

$$f'(n) - f'(n-1) = q^{n+1}f'(n-2). \tag{4.9}$$

We now define

$$L_n(q) = F_{n+2}(q) - q^n F'_{n-2}(q) \qquad (n\geqslant 2), \tag{4.10}$$

$$F'_{n+1}(q) = f'(n-1), \qquad F'_0(q) = 0. \tag{4.11}$$

We have

$$F'_{n+1}(q) - F'_n(q) = q^{n+1}F'_{n-1}(q); \tag{4.12}$$

this recurrence should be compared with (3.4).

The first few values of $L_n(q)$ are

$$L_2(q) = 1+q+q^2, \qquad L_3(q) = 1+q+q^2+q^3,$$
$$L_4(q) = 1+q+q^2+q^3+2q^4+q^6,$$
$$L_5(q) = 1+q+q^2+q^3+2q^4+2q^5+q^6+q^7+q^8.$$

It follows from (4.8) that

$$\sum_{n=0}^{\infty} F'_{n+1}(q)x^n = \sum_{k=0}^{\infty} q^{(k+1)^2}x^{2k}/(x)_{k+1}. \tag{4.13}$$

The first few values of $F'_n(q)$ are

$$F'_1(q) = q, \qquad F'_2(q) = q, \qquad F'_3(q) = q+q^4, \qquad F'_4(q) = q+q^4+q^5,$$
$$F'_5(q) = q+q^4(1+q+q^2)+q^9, \qquad F'_6(q) = q+q^4(1+q+q^2+q^3)+q^9(1+q+q^2).$$

Thus, for example

$$L_4(q) = F_6(q) - q^4F'_2(q) = (1+q+q^2+q^3+2q^4+q^5+q^6) - q^5,$$
$$L_5(q) = F_7(q) - q^5F'_3(q) = (1+q+q^2+q^3+2q^4+2q^5+2q^6+q^7+q^8+q^9) - q^5(q+q^4),$$

in agreement with the values previously found.

It would be of interest to find a simple combinatorial interpretation of $F'_n(q)$.

5. By means of the recurrence (3.4) we can define $F_n(q)$ for negative n. Put

$$\bar{F}_n(q) = (-1)^{n-1} F_{-n}(q)\,. \tag{5.1}$$

Then (3.4) becomes

$$\bar{F}_n(q) = q^n(\bar{F}_{n-1}(q) + \bar{F}_{n-2}(q)) \qquad (n \geq 2)\,. \tag{5.2}$$

where

$$\bar{F}_0(q) = 0, \qquad \bar{F}_1(q) = q\,.$$

Put

$$\Phi(x) = \sum_{n=0}^{\infty} \bar{F}_n(q)x^n\,. \tag{5.3}$$

Then

$$\Phi(x) = qx + \sum_{n=2}^{\infty} q^n(\bar{F}_{n-1}(q) + \bar{F}_{n-2}(q))x^n\,,$$

so that

$$\Phi(x) = qx + qx(1+qx)\Phi(qx)\,. \tag{5.4}$$

Thus

$$\begin{aligned}\Phi(x) &= qx + qx(1+qx)\left\{q^2x + q^2x(1+q^2x)\Phi(q^2x)\right\}\\ &= qx + q^3x^2(1+qx) + q^3x^2(1+qx)(1+q^2x)\Phi(q^2x)\,.\end{aligned}$$

At the next stage we get

$$\Phi(x) = qx + q^3x^2(1+qx) + q^6x^3(1+qx)(1+q^2x) + q^6x^3(1+qx)(1+q^2x)(1+q^3x)\Phi(q^3x)\,.$$

The general formula is evidently

$$\Phi(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+1)(k+2)}x^{k+1}(1+qx)(1+q^2x)\cdots(1+q^kx)\,. \tag{5.5}$$

Since

$$(1+qx)(1+q^2x)\cdots(1+q^kx) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)}x^j\,,$$

(5.5) becomes

$$\Phi(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}(k+1)(k+2)}x^{k+1} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)}x^j = \sum_{n=0}^{\infty} x^{n+1} \sum_{2j \leq n} \begin{bmatrix} n-j \\ j \end{bmatrix} q^{\frac{1}{2}j(j+1)+\frac{1}{2}(n-j+1)(n-j+2)}\,.$$

Comparison with (5.3) gives

$$\bar{F}_{n+1}(q) = \sum_{2j \leq n} \begin{bmatrix} n-j \\ j \end{bmatrix} q^{\frac{1}{2}(n+1)(n+2)-nj+j(j-1)}\,. \tag{5.6}$$

The first few values of $\bar{F}_n(q)$ are

$$\bar{F}_2(q) = q^3, \qquad \bar{F}_3(q) = q^4(1+q^2), \qquad \bar{F}_4(q) = q^7(1+q+q^3)\,,$$
$$\bar{F}_5(q) = q^9(1+q^2+q^3+q^4+q^6), \qquad \bar{F}_6(q) = q^{13}(1+q+q^2+q^3+q^4+q^5+q^6+q^8)\,.$$

REFERENCES

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