ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-292 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain and prove a formula for the number $S(n,t)$ of terms in

$$(x_1 + x_2 + \cdots + x_n)^t ,$$

where n and t are integers with $n > 0$ and $t \geq 0$.

B-293 Proposed by Harold Don Allen, Nova Scotia Teachers College, N.S., Canada.

Identify $T, W, H, R, E, F, I, V,$ and G as distinct digits in $\{1, 2, \cdots, 9\}$ such that we have the following sum (in which 1 and 0 are the digits 1 and 0):

$$\begin{array}{r} 1 \\ 1 \\ TWO \\ THREE \\ \underline{FIVE} \\ EIGHT \end{array}$$

B-294 Proposed by Richard Blazej, Queens Village, New York.

Show that $F_n L_k + F_k L_n = 2F_{n+k}$ for all integers n and k.

B-295 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Find a closed form for

$$\sum_{k=1}^{n} (n+1-k)F_{2k} = nF_2 + (n-1)F_4 + \cdots + F_{2n} .$$

B-296 Proposed by Gary Ford, Vancouver, British Columbia, Canada.

Find constants a and b and a transcendental function G such that

$$G(y_{n+3}) + G(y_n) = G(y_{n+2})G(y_{n+1})$$

whenever y_n satisfies

$$y_{n+2} = ay_{n+1} + by_n .$$

B-297 Proposed by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

Obtain a recursion formula and a closed form in terms of Fibonacci and Lucas numbers for the sequence $\{G_n\}$ defined by the generating function:

$$(1-3x-x^2+5x^3+x^4-x^5)^{-1} = G_0+G_1x+G_2x^2+\cdots+G_nx^n+\cdots .$$

SOLUTIONS

FIBONACCI COMPLEX NUMBERS

B-268 Proposed by Warren Cheves, Littleton, North Carolina.

Define a sequence of complex numbers $\{C_n\}$, $n = 1, 2, \cdots$, where $C_n = F_n + iF_{n+1}$. Let the conjugate of C_n be $\overline{C}_n = F_n - iF_{n+1}$. Prove

(a) $$C_n\overline{C}_n = F_{2n+1}\,;$$

(b) $$C_n\overline{C}_{n+1} = F_{2n+2}+(-1)^n i.$$

Solution by J. L. Hunsucker, University of Georgia, Athens, Georgia.

In solving this problem we quote identities by number from V.E. Hoggatt's *Fibonacci and Lucas Numbers.*

First

$$C_n\overline{C}_n = F_n^2+(F_{n+1})^2 = F_{2n+1}$$

by I_{11} in Hoggatt and (a) is proved. Second,

$$C_n\overline{C}_{n+1} = (F_nF_{n+1}+F_{n+1}F_{n+2})+i(F_{n+1}^2-F_nF_{n+2})\,.$$

Then

$$F_nF_{n+1}+F_{n+1}F_{n+2} = F_{n+1}(F_n+F_{n+2}) = (F_{n+2}-F_n)(F_{n+2}+F_n) = F_{n+2}^2-F_n^2 = F_{2n+2}$$

by I_{10}. Also, by I_{13},

$$F_{n+1}^2-F_nF_{n+2} = (-1)(-1)^{n+1} = (-1)^n\,,$$

and (b) is proved.

Also solved by Wray G. Brady, Herta T. Freitag, Ralph Garfield, John W. Milsom, C.B.A. Peck, M.N.S. Swamy, P. Thrimurthy, Gregory Wulczyn, David Zeitlin, and the Proposer.

DIAGONALIZING THE Q MATRIX

B-269 Proposed by Warren Cheves, Littleton, North Carolina.

Let Q be the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of Q are α and β, where

$$\alpha = (1+\sqrt{5})/2 \qquad \text{and} \qquad \beta = (1-\sqrt{5})/2.$$

Since the eigenvalues of Q are distinct, we know that Q is similar to a diagonal matrix A Show that A is either

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}.$$

Solution by P. Thrimurthy, Gujarat University, Ahmedabad, India.

The eigenvectors corresponding to the two eigenvalues α and β are

$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \beta \\ 1 \end{pmatrix},$$

respectively. Hence the transforming matrix is either

$$P = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \qquad \text{or} \qquad R = \begin{pmatrix} \beta & \alpha \\ 1 & 1 \end{pmatrix}.$$

Now

$$P^{-1}QP = \begin{pmatrix} 1/(\alpha-\beta) & -\beta/(\alpha-\beta) \\ -1/(\alpha-\beta) & \alpha/(\alpha-\beta) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad R^{-1}QR = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} .$$

Also solved by David Zeitlin and the Proposer.

A MULTIPLE OF L_{2m+1}

B-270 Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refute the following: If k is odd,

$$L_k \big| [F_{(n+2)k} - F_{nk}] .$$

Solution by C.B.A. Peck, State College, Pennsylvania.

$$F_{(n+2)k} - F_{nk} = L_k F_{(n+1)k}$$

for k odd (see *The Fibonacci Quarterly,* Vol. 7, No. 5 (Dec. 1969), p. 486).

Also solved by W.G. Brady, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIND THE MULTIPLE OF $L_{2m} - 2$

B-271 Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refute the following: If k is even, $L_k - 2$ is an exact divisor of:

(a) $$F_{(n+2)k} + 2F_k - F_{nk} \;;$$

(b) $$F_{(n+2)k} - 2F_{(n+1)k} + F_{nk} \,;$$

and

(c) $$2[F_{(n+2)k} - F_{(n+1)k} + F_k] .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Since

$$F_{(n+2)k} - L_k F_{(n+1)k} + (-1)^k F_{nk} = 0 ,$$

$$F_{(n+2)k} - 2F_{(n+1)k} + (-1)^k F_{nk} = (L_k - 2)F_{(n+1)k} .$$

For k even,

$$(L_k - 2) | (F_{(n+2)k} - 2F_{(n+1)k} + F_{nk})$$

and (b) is true. (a) False. For $n = 0$,

$$(L_k - 2) \nmid (F_{2k} + 2F_k)$$

when $k = 4$. (c) False. For $n = 0$,

$$(L_k - 2) \nmid 2F_{2k}$$

when $k = 4$.

Also solved by C.B.A. Peck, Gregory Wulczyn, and the Proposer.

A NONLINEAR RECURRENCE

B-272 Proposed by Gary G. Ford, Vancouver, British Columbia, Canada.

Find at least some of the sequences $\{ y_n \}$ satisfying

$$y_{n+3} + y_n = y_{n+2} y_{n+1} .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Three solutions are given by:

(1) $$y_n = 0 \text{ for all } n.$$

(2) $$y_n = 2 \text{ for all } n.$$

(3) Let b denote a parameter, independent of n. Then one may let $y_{4m} = b$, $y_{4m+1} = -1 = y_{4m+3}$, $y_{4m+2} = 1 - b$, for all integers m.

NOTE: Herta T. Freitag and P. Thrimurthy each pointed out that any three consecutive terms may be chosen arbitrarily, and then the recurrence determines the other terms. Another version of this problem is proposed in this issue as B-296.

GOLDEN MINIMUM PERIMETER

B-273 Proposed by Marjorie Bicknell, A.C. Wilcox High School, Santa Clara, California.

Construct any triangle $\triangle ABC$ with vertex angle $A = 54°$ and median $\overline{AM}$ to the side opposite A such that $AM = 1$. Now, inscribe $\triangle XYM$ in $\triangle ABC$ so that M is the midpoint of $\overline{BC}$, and X and Y lie between A and B and between A and C, respectively. Find the minimum perimeter possible for the inscribed triangle, $\triangle XYM$.

Solution by the Proposer.

Construct $\angle MAB \cong \angle M'AB$, $AM' = AM$; $\angle MAC \cong \angle M''AC$, $AM'' = AM$. Then draw $M'M''$, intersecting $\overline{AB}$ at X and $\overline{AC}$ at Y. Since, by S.A.S., $\triangle M'AX \cong \triangle MAX$ and $\triangle M''AY \cong \triangle MAY$, $MX + XY + YM = M'X + XY + YM''$, which is a minimum when M', X, Y, and M'' are colinear. So, the minimum perimeter is given by the length $M'M''$. Also, $\angle M'AM'' = 2A$. (This construction was given by Samuel L. Greitzer, Rutgers University, as solution to a problem appearing in *Summation*, Association of Teachers of Mathematics of New York City, Spring, 1972.)

By the Law of Cosines,

$$(M'M'')^2 = (AM')^2 + (AM'')^2 - 2(AM')(AM'')\cos 2A = 2(AM)^2(1-\cos 2A) = 4(AM)^2\sin^2 A.$$

Thus, $M'M'' = 2(AM)(\sin A)$.

Now, that $\sin 54° = (1+\sqrt{5})/4 = \phi/2$ is easily seen from the following:

$$\sin 36° = \cos 54° = \cos 18° \cos 36° - \sin 18° \sin 36°$$

$$2\cos 18° \sin 18° = \cos 18° \cos 36° - 2\sin^2 18° \cos 18°$$

leading to

$$4\sin^2 18° + 2\sin 18° - 1 = 0$$

so that

$$\sin 18° = (\sqrt{5}-1)/4 = 1/2\phi .$$

Then,

$$\cos 36° = 1 - 2\sin^2 18° = (1+\sqrt{5})/4 = \phi/2 = \sin 54° .$$

Therefore, the minimum perimeter is given by

$$M'M'' = 2(AM)(\sin 54°) = 2(1)(\phi/2) = \phi ,$$

the Golden Section Ratio.

Notice that nowhere was the fact that $\overline{AM}$ was a median required. If M is any point between B and C such that $AM = 1$, we have the same minimum perimeter.

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