SOME SEQUENCES GENERATED BY SPIRAL SIEVING METHODS

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The object of this note is to point out some curious sequences which may be generated by natural number spirals and rotating grids. The method is a combination of the spiral introduced by Ulam [2] in his studies of prime number distribution and a well known technique employed in cryptographic work. We illustrate with Fibonacci numbers.

Ulam considers a spiral numbering of the lattice points in the plane, i.e., by starting at (0,0) and proceeding counterclockwise in a spiral so that

$$(0,0) \to 1,\quad (1,0) \to 2,\quad (1,1) \to 3,\quad (0,1) \to 4,\quad (-1,1) \to 5,\quad (-1,0) \to 6,\quad (-1,-1) \to 7,\quad (0,-1) \to 8,$$
$$(1,-1) \to 9,\quad (2,-1) \to 10,\quad (2,0) \to 11,\quad (2,1) \to 12,\quad (0,2) \to 13,\quad (-1,2) \to 14,\quad etc.$$

This mapping gives us the spiral below.

(1)

17	16	15	14	13	
18	5	4	3	12	
19	6	1	2	11	
20	7	8	9	10	27
21	22	23	24	25	26

A nice illustration of the basic Ulam spiral makes up the front cover of the March 1964 *Scientific American.* In the same issue Martin Gardner [1] gives an account of Ulam's work. Briefly, Ulam marks the primes $(1,2,3,5,7,11,\cdots)$ in the spiral and studies the visual display for patterns or almost-patterns in the prime number sequence. By use of a computer at Los Alamos he is able to generate displays having around 65,000 points in them. It would be of interest to try something of the same sort for the Fibonacci, Lucas, and other recurrent sequences, however the writer does not have available such versatile equipment as that used by Ulam and his colleagues at Los Alamos, so we have little to suggest about possible patterns in a spiral display of Fibonacci numbers. Of course, the fact that we now know [3], [4] that 1 and 144 are the only square Fibonacci numbers does tell us that the diagonals $1, 9, 25, 49, \cdots$ and $4, 16, 36, 64, \cdots$ will be conspicuously blank in such a display.

Now, there is a technique in cryptographic work which makes use of a rotating grid. We can best illustrate by means of an example. Consider the message, "INTUITION LIKE A FLASH OF LIGHTNING LASTS ONLY FOR A SECOND." We write this in a square array

I N T U I T I
O N L I K E A
F L A S H O F
L I G H T N I
N G L A S T S
O N L Y F O R
A S E C O N D

and then impose a prepunched grid, e.g., of the form (where an X indicates a hole)

(2)

X		X			X	
			X			
		X			X	
			X	X		
						X
X		X			X	
			X			

and copy out the visible letters, which are (serially, row by row) ITTIAOHTSOLOC. We then rotate the grid counterclockwise through 90° and again copy out the visible letters, which are IOLESIHINLTAN. Two more rotations gives us UNKAFGHGSYSOD and NILHFLHNANFRE. Running these four groups together and breaking the whole up into convenient blocks then gives us the enciphered message. To decipher, one merely places the grid on a sheet of paper, writes in the letters serially, row by row, thirteen at a time here, rotating the grid until all four positions are used, removes the grid and reads off the message. Here we have used a 7 by 7 grid which leaves the middle point fixed (H). This is unsatisfactory for cryptographic work in some cases and most ordinary uses involve an even-order grid.

The effect of an odd-order grid in the case of superposition on the natural number spiral is to partition the natural numbers into four sets, any two of which have only the number 1 in common.

It is clear that the very special cryptographic grid cannot be made from the Fibonacci sequence (or the prime number sequence) without adding and/or deleting elements, since any given square annulus of the grid must be so designed that one-fourth of its lattice points are punched, and in such a way that the same hole does not appear under successive rotations of 90° until the original position is assumed. We shall not discuss how this can be effected.

We modify the rotating grid as follows. On the original natural number spiral (1) superimpose a square sheet of paper which will just cover the first $(2n-1)^2$ natural numbers, unity being kept at the center. Make a grid by punching the sheet wherever an element a_k $(k = 1, 2, 3, \cdots)$ of a given sequence appears in the natural number spiral. We shall call this the (counterclockwise) spiral grid of the sequence $\{a_k\}$. We next rotate the spiral grid through 90° and read off from the natural number spiral a new sequence generated by the spiral grid of our original sequence. With any given sequence there will be associated three new sequences, and by turning the grid over (making it a clockwise spiral grid) we can generate four other sequences. Clearly all these eight sequences will be somehow related.

For a grid measuring $2n-1$ by $2n-1$ $(n \geqslant 2)$ there will be the natural numbers from 1 through $(2n-1)^2$ with the outer square annulus containing the successive natural numbers from $(2n-3)^2+1$ to $(2n-1)^2$. If an element a_k of our given sequence lies in the outer square annulus, then so will the corresponding element b_k^i of any of the associated sequences obtained by use of the grid. It is possible to work out complicated formulas relating b_k to a_k depending upon the position of an element in the annulus. For example, any two diagonally opposite elements in the outer annulus have numerical difference $4n$.

We give below, in Table 1, a few values for the sequences generated by the counterclockwise spiral grid of the Fibonacci sequence (I, II, III, IV) and also for the clockwise grid (I′, II′, III′, IV′).

Here, $d = d_k$ is the minimum positive difference between terms in the sequences, or

$$d = d_k = \min_{i,j}\,(b_k^i - b_k^j) > 0,$$

$$\text{(with } d_0 \text{ def.} = 0)$$

for Counterclockwise (I – IV), or for Clockwise (I′ – IV′).

In our table, $a_k = F_{k+1}$, with

$$F_{k+1} = F_k + F_{k-1}, \qquad F_0 = 0, \qquad F_1 = 1.$$

It is convenient to begin our sequence with F_2 instead of making some rules about how to interpret $0, 1, 1, 2, 3, \cdots$. (The indistinguishability of F_1 and F_2 prevents us from calling the ordinary Fibonacci sequence a subset of the set of all natural numbers.)

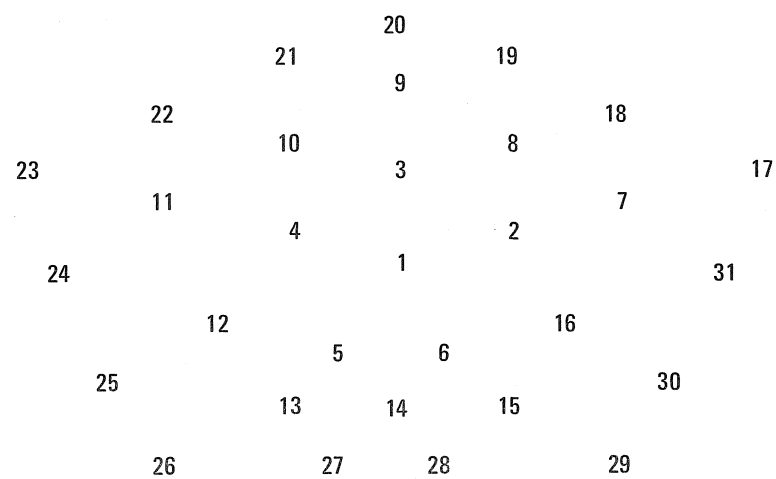
There is no reason to confine our attention to spirals based on a square. Ulam's work with the sequence of primes quite naturally fits in well with such a spiral because quadratic polynomials $Ax^2 + Bx + C$ are often so rich in

Table 1

	a_k	b_k^2	b_k^3	··· etc.					
k	I	II	III	IV	I′	II′	III′	IV′	d
1	1	1	1	1	1	1	1	1	0
2	2	2	4	3	3	2	2	3	1
3	3	4	6	6	5	5	4	4	1
4	5	5	7	8	6	7	7	6	1
5	8	7	9	9	8	8	9	9	1
6	13	17	13	17	17	13	17	13	4
7	21	25	21	25	25	21	25	21	4
8	34	40	46	28	34	40	46	28	6
9	55	63	71	79	67	75	51	59	8
10	89	99	109	119	103	113	83	93	10
11	144	156	168	132	134	146	158	122	12
12	233	249	265	281	265	281	233	249	16
13	377	397	417	437	405	425	365	385	20

primes for integral values of x (Euler's polynomial x^2+x+41 being the most well-known example). However, to exhibit other properties of a sequence, as well as to generate variations of a given sequence, it is natural to pass on to figurate numbers as the basis of our spirals. That is, we may consider a polygon of m sides.

Consider, for example, a pentagonal spiral as shown below.



It would be of interest to examine the distribution of primes, Fibonacci numbers, etc., in an extended pentagon with thousands of points, and of course this would require quite an elaborate computer set-up.

It is fairly easy to type out a pentagonal spiral on ordinary typing paper with 456 points and this is sufficient to give an idea of how the pentagonal spiral grid of the Fibonacci sequence can be used to generate curious sequences. Here, of course, we shall have in all ten sequences. The sequences are tabulated below in Table 2.

The number d tabulated in the last column is defined as before by

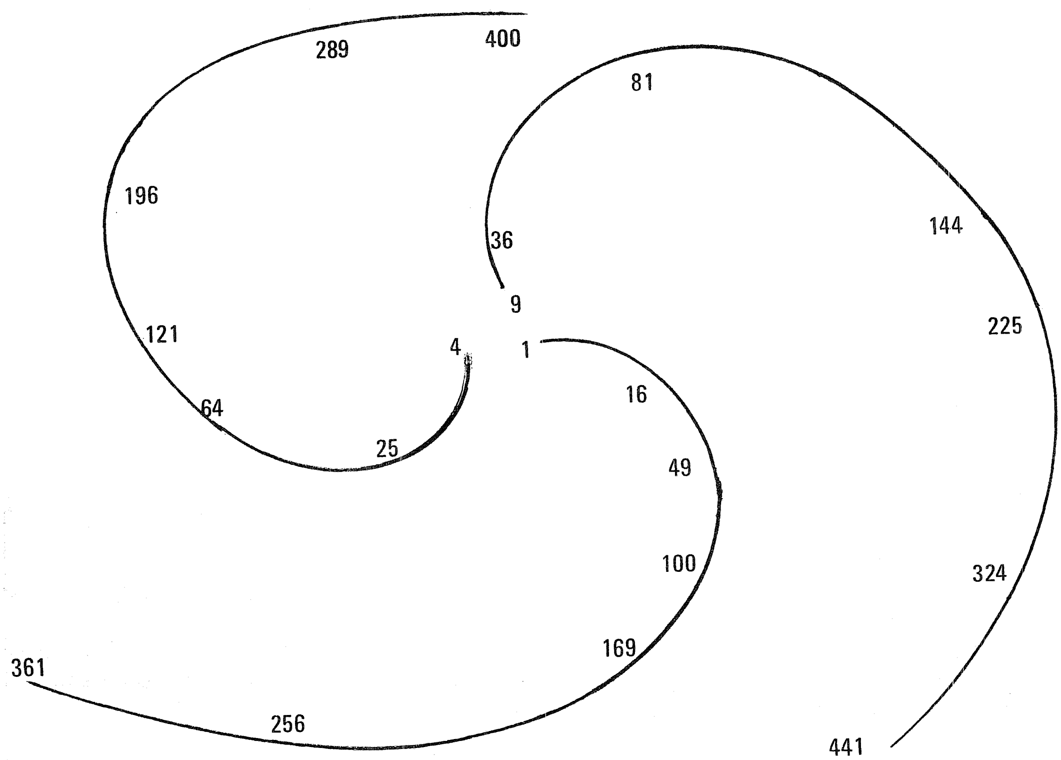
$$d = \min_{i,j}\,(b_k^i - b_k^j) > 0 \tag{3}$$

(for I – V or I′ – V′), and it is not difficult to see that for any given value of k the numbers II – V determined by our grid will differ from the Fibonacci number a_k by a multiple of the number d. The reader may find it of interest to try and develop a general formula for d in terms of k and m (generalizing to an m-gon).

Table 2

	a_k	b_k	b_k	… etc.							
k	I	II	III	IV	V	I′	II′	III′	IV′	V′	d
1	1	1	1	1	1	1	1	1	1	1	0
2	2	3	2	3	2	3	2	3	2	2	1
3	3	4	4	5	4	4	4	5	4	3	1
4	5	6	5	6	6	6	5	6	6	5	1
5	8	10	7	9	11	10	7	9	11	8	1
6	13	15	12	14	16	15	12	14	16	13	1
7	21	24	27	30	18	19	22	25	28	31	3
8	34	38	42	46	50	38	42	46	50	34	4
9	55	60	65	70	75	59	64	69	74	54	5
10	89	95	101	77	83	77	83	89	95	101	6
11	144	152	160	168	176	156	164	172	180	148	8
12	233	243	253	263	273	241	251	261	271	231	10
13	377	389	341	353	365	371	383	335	347	359	12

The visual display of *perfect squares* in a pentagonal spiral turns out to be a simple *trefoil* spiral appearing somewhat as diagrammed below.



This is easily verified to be in accord with the fact that the three arms of the spiral are formed by squares of form $(3n)^2$, $(3n+1)^2$, and $(3n+2)^2$, respectively.

Finally, we turn to the case of a *triangular* spiral grid. Because of the hexagonal rotational character in this case, one may generate 12 sequences for a given spiral grid, 6 counterclockwise and 6 clockwise. A portion of the triangular spiral appears below.

39		7		35	
	15		13		50
26		3		22	
	8		6		34
16		1		12	
	4		2		21
9		10		5	
	18		19		11

The 12 sequences generated by a triangular spiral grid based on the Fibonacci numbers are tabulated in Table 3.

Table 3

I	II	III	IV	V	VI	I′	II′	III′	IV′	V′	VI′	d	d*
1	1	1	1	1	1	1	1	1	1	1	1	0	0
2	4	3	2	2	3	3	3	2	4	2	2	1	1
3	6	4	8	4	6	4	8	4	6	3	6	1	2
5	8	7	10	6	10	6	10	5	10	7	8	1	2
8	15	10	18	9	12	9	16	8	19	10	13	1	3
13	22	16	26	19	30	15	29	18	21	12	25	3	4
21	23	25	27	29	31	27	30	31	22	23	26	4	4
34	35	39	40	44	45	40	44	45	34	35	39	5	5
55	57	61	63	49	51	51	55	57	61	63	49	6	6
89	70	97	77	105	84	99	81	107	67	91	74	8	7

Here, d is based on either I – III – V or I′ – III′ – V′ while d^* is based on II – IV – VI or II′ – IV′ – VI′. This is because II, IV, and VI arise from the hexagonal effect. Thus it seems of interest to list d as based on both triangular pattern and hexagonal.

With this much as an introduction to the notion of a spiral grid for generating variations of a given sequence, we shall close this account. Our purpose has been mainly to exhibit the results of some calculations and suggest possible avenues of research. Various questions could be posed. For example: What can be said about divisibility properties of the new sequences? What can be said about when such sequences will satisfy simple recurrence relations? Does any of this shed light on when a Fibonacci number may be a figurate number? Can simple formulas be written for the various derived sequences? What is a simple formula for the number we have called d?

REFERENCES

1. Martin Gardner, "Mathematical Games, The Remarkable Lore of the Prime Numbers," *Scientific American,* 210 (1964), March, No. 3, 120 *et seq.*
2. M.L. Stein, S.M. Ulam, and M.B. Wells, "A Visual Display of Some Properties of the Distribution of Primes," *Amer. Math. Monthly,* 71 (1964), pp. 516–520.
3. Oswald Wyler, Solution of Problem 5080, *Amer. Math. Monthly,* 71 (1964), pp. 220–222.
4. John H.E. Cohn, "Square Fibonacci Numbers, etc.," *The Fibonacci Quarterly,* Vol. 2, No. 2 (April, 1964), pp. 109–113.

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