SOME PROPERTIES OF A FUNDAMENTAL RECURSIVE SEQUENCE OF ARBITRARY ORDER

A. G. SHANNON
The New South Wales Institute of Technology, Broadway, Australia

1. INTRODUCTION

In this paper, three properties of a fundamental recursive sequence of arbitrary order are examined by analyzing and recombining the zeros of the associated auxiliary equation. The three properties in question are Simson's relation (Sections 2, 3, 4), a Lucas identity discussed by Jarden (Section 5), and Horadam's Pythagorean triples (Section 6).

We define a fundamental i^{th} order linear recursive sequence $\{u_n^{(i)}\}$ in terms of the linear recurrence relation

$$(1.1)\qquad \begin{aligned} u_n^{(i)} &= \sum_{r=1}^{i} P_{ir}u_{n-r}^{(i)} \qquad n > 0,\\ u_n^{(i)} &= 1 \qquad n = 0,\\ u_n^{(i)} &= 0 \qquad n < 0, \end{aligned}$$

in which the P_{ir} are arbitrary integers.

The "fundamental" character of this sequence has been shown elsewhere by the present writer [7].

Associated with the recurrence relation in (1.1) is the auxiliary equation

$$(1.2)\qquad f_i(x) \equiv \prod_{r=1}^{j} (x - a_{ir}) = 0$$

in which it is assumed that the complex numbers a_{ir} are distinct. We shall restrict ourselves to this non-degenerate case, but the basic arguments survive when the zeros of (1.2) are not distinct. In the degenerate case the order of the i-related sequence described below may be reduced.

We define an "i-related sequence of order j," $\{x_n^{(j)}\}$, as one which satisfies the j^{th} order recurrence relation

$$(1.3)\qquad \begin{aligned} x_n^{(j)} &= \sum_{r=1}^{j} (-1)^{r+1} Q_{ir}x_{n-r}^{(j)} \qquad n > 0,\\ x_n^{(j)} &= 1 \qquad n = 0, \qquad j = \binom{i}{2},\\ x_n^{(j)} &= 0 \qquad n < 0, \end{aligned}$$

with an auxiliary equation

$$(1.4)\qquad g_j(x) \equiv \prod_{\substack{r=1\\ r<m}}^{j} (x - a_{ir}a_{im}) = 0,$$

in which the Q_{ir} are integers and where the $a_{ir}a_{im}$ are the zeros of (1.2). For example, when $i = 3$, $j = 3$, and

$$f_3(x) = (x - a_{31})(x - a_{32})(x - a_{33})$$

$$g_3(x) = (x - a_{31}a_{32})(x - a_{31}a_{33})(x - a_{32}a_{33}) = x^3 - \Sigma a_{31}a_{32}x^2 + \Sigma a_{31}^2 a_{32}a_{33}x - (a_{31}a_{32}a_{33})^2.$$

We choose the symbol Q_{ir} rather than Q_{jr} because the Q_{ir} can be expressed in terms of the a_{ir} as we show in Equation (4.7).

2. SIMSON'S RELATION

For the fundamental sequence of Lucas [4], $\{u_n^{(2)}\}$, (in our notation), Simson's relation takes the form

$$(u_n^{(2)})^2 - (u_{n-1}^{(2)})(u_{n+1}^{(2)}) = (a_{21}a_{22})^n = x_n^{(1)} \tag{2.1}$$

since $\binom{2}{2} = 1$.

More generally we assert that

$$(u_n^{(i)})^2 - (u_{n-1}^{(i)})(u_{n+1}^{(i)}) = x_n^{(j)}, \qquad j = \binom{i}{2}. \tag{2.2}$$

To prove this we use the fact that

$$u_n^{(i)} = \sum_{r=1}^{i} A_{ir}a_{ir}^n, \tag{2.3}$$

wherein the A_{ir} are determined by the initial values of $u_n^{(i)}$, $n = 0, 1, \cdots, i-1$. Thus the left-hand side of (2.2) becomes, after cancellation of terms,

$$-\sum_{r<m} A_{ir}A_{im}(a_{ir}-a_{im})^2(a_{ir}a_{im})^{n-1} = \sum_{r<m} B_{js}\beta_{js}^n$$

in which

$$\beta_{js} = a_{ir}a_{im}, \qquad \text{and} \qquad B_{js}\beta_{js} = -A_{ir}A_{im}\times(a_{ir}-a_{im})^2.$$

Note that $j = \binom{i}{2}$ since there are ia_{ir} to be taken two at a time. Note further that $A_{ir}A_{im}$ contains $(a_{ir}-a_{im})^2$ in its denominator; see Jarden [3, p. 107].

The result (2.2) does not tell us much about the specific terms of $\{x_n^{(j)}\}$. We can find the initial terms by substituting successively the first $j+1$ values of $\{u_n^{(i)}\}$ in (2.2). For example, the first three terms can be found as follows:

$$(u_0^{(i)})^2 - (u_{-1}^{(i)})(u_1^{(i)}) = 1 = x_0^{(j)}.$$

$$(u_1^{(i)})^2 - (u_0^{(i)})(u_2^{(i)}) = P_{i1}^2 - P_{i1}^2 - P_{i2} = \sum_{r<m} a_{ir}a_{im}$$

$$= Q_{i1} = Q_{i1}x_0^{(j)} = x_1^{(j)}.$$

$$(u_2^{(i)})^2 - (u_1^{(i)})(u_3^{(i)}) = P_{i2}^2 - P_{i1}P_{i3} = Q_{i1}x_1^{(j)} - Q_{i2}x_0^{(j)} = x_2^{(j)}.$$

One can examine the nature of $\{x_n^{(j)}\}$ by the use of the multinomial expression for $u_n^{(i)}$, namely,

$$u_n^{(i)} = \sum_{\Sigma r\lambda_r = n} \frac{(\Sigma\lambda)!}{\lambda_1!\lambda_2!\cdots\lambda_n!} \prod_{r=1}^{i} P_{ir}^{\lambda_r}, \tag{2.4}$$

and we shall do that in Section 4. We first consider the auxiliary equation for $\{x_n^{(j)}\}$ and the coefficient, Q_{ir}, of the recurrence relation separately.

Equation (2.4) follows if we adapt Macmahon [5, pp. 2–4], because $u_n^{(i)}$ is in fact the homogeneous product sum of weight n of the quantities a_{ij}. It is the sum of a number of symmetric functions formed from a partition of the number n. The first three cases are

$$u_1^{(i)} = P_{i1} = \sum a_{i1},$$

$$u_2^{(i)} = P_{i1}^2 + P_{i2} = \sum a_{i1}^2 + \sum a_{i1}a_{i2},$$

$$u_3^{(i)} = P_{i1}^3 + 2P_{i1}P_{i2} + P_{i3} = \sum a_{i1}^3 + \sum a_{i1}^2 a_{i2} + \sum a_{i1}a_{i2}a_{i3}.$$

In general,

$$u_n^{(i)} = \sum_{\Sigma\lambda=n} a_{i1}^{\lambda_1}a_{i2}^{\lambda_2}\cdots = \sum_{\Sigma\lambda=n}\prod_{r=1}^{i} a_{ir}^{\lambda_r}.$$

It is of interest to note that another formula for $u_n^{(i)}$ can be given by

$$(2.5)\qquad u_n^{(i)} = \sum_{r=1}^{i} a_{ir}^{i+n-1} \Big/ \prod_{r\neq s} (a_{ir} - a_{is}) .$$

From Jarden [3, p. 107] we have that

$$(2.6)\qquad u_n^{(i)} = \sum_{r=1}^{i} a_{ir}^{n} D_r/D \;,$$

where D is the Vandermonde of the roots given by

$$(2.7)\qquad D = \sum_{r=1}^{i} a_{ir}^{i-1} \prod_{\substack{r\neq s\neq t\\ s<t}} (a_{it} - a_{is}) = \prod_{r>s} (a_{ir} - a_{is}) \prod_{s<t} (a_{it} - a_{is})$$

and D_r is the determinant of order i obtained from D on replacing its r^{th} column by the initial terms of $\{u_n^{(i)}\}$, $n = 0, 1, 2, \cdots, i-1$. It thus remains to prove that

$$(2.8)\qquad D_r = a_{ir}^{i-1} \prod_{\substack{r\neq s\neq t\\ s<t}} (a_{it} - a_{is}) = Da_{ir}^{i-1} \Big/ \prod_{r>s} (a_{ir} - a_{is}) .$$

We use the method of the contrapositive. If

$$D_r \neq a_{ir}^{i-1} \prod_{\substack{r\neq s\neq t\\ s<t}} (a_{it} - a_{is}) ,$$

then

$$D = \sum_{r=1}^{i} D_r \qquad \text{(from (2.6) with } n = 0)$$

$$\neq \sum_{r=1}^{i} a_{ir}^{i-1} \prod_{\substack{r\neq s\neq t\\ s<t}} (a_{it} - a_{is})$$

which contradicts (2.7). This proves (2.8) and we have established that

$$u_n^{(i)} = \sum_{r=1}^{i} a_{ir}^{n} D_r/D = \sum_{r=1}^{i} a_{ir}^{i+n-1} D_r/Da_{ir}^{i-1} = \sum_{r=1}^{i} a_{ir}^{i+n-1} / \prod_{r>s} (a_{ir} - a_{is}) ,$$

as required.

3. AUXILIARY EQUATIONS

van der Poorten [6] has proved that if $f(x)$ is a polynomial with complex coefficients, and $\{U_n\}$, $\{V_n\}$ denote sequences of elements of C, and if

$$\prod_{r=1}^{i} (E - a_r)U_n = 0, \qquad f(E)V_n = 0 ,$$

then

$$h(E)U_nV_n = 0, \qquad n \geqslant 0 ,$$

where E is the operator on sequences which performs the action

$$EU_n = V_{n+1} , \qquad EV_n = V_{n+1}, \cdots \qquad n \geqslant 0 ,$$

and $H(x)$ denotes the monic polynomial which is the least common multiple of the polynomials

$$f(x/a_1),\ f(x/a_2),\ \cdots,\ f(x/a_i)\ ,$$

in which it is assumed that $a_1, a_2, \cdots, a_i$ are non-zero and distinct.

We now consider $\Pi(E - a_{ir})u_n^{(i)} = 0$ in place of both $\Pi(E - a_r)U_n$ and $f(E)V_n$. Then it follows from above that

$$h(E)(u_n^{(i)})^2 = 0\ , \tag{3.1}$$

where $H(x)$ is the l.c.m. of

$$\prod_{s,r=1}^{i} ((x/a_{is}) - a_{ir})$$

which can be re-written as $P_{ii}^{-1}\Pi(x - a_{ir}a_{is})$ since

$$P_{ii} = \prod_{s=1}^{i} a_{is}\,.$$

Thus the zeros of $h(x)$ are $a_{i1}, \cdots, a_{ii}$ taken 2 at a time.

In (3.1) we have established that the sequence

$$\left\{(u_n^{(i)})^2\right\}$$

satisfies a linear recurrence relation of order $\binom{i+1}{2}$ with auxiliary equation

$$F_{i+j}(x) = \prod_{\lambda_1+\lambda_2=2} (x - a_{ir}^{\lambda_1} a_{im}^{\lambda_2})\ , \tag{3.2}$$

where $j = \binom{i}{2}$ as before since

$$\binom{i+1}{2} = \binom{i}{1} + \binom{i}{2} = i + j\,.$$

Note that r may equal m in (3.2), and so

$$F_{i+j}(x) = \prod_{r=1}^{i} (x - a_{ir}^2) \prod_{m<s} (x - a_{im}a_{is})\,.$$

If we let

$$F_i(x) = \prod_{r=1}^{i} (x - a_{ir}^2)\,, \tag{3.3}$$

which is the auxiliary equation associated with the sequence $\left\{s_{2n}^{(i)}\right\}$, then we have proved

$$g_j(x) = F_{i+j}(x)/F_i(x)\ . \tag{3.4}$$

The auxiliary equation for $\left\{x_n^{(j)}\right\}$ can also be represented in terms of the coefficients of the corresponding recurrence relation by

$$g_j(x) = x^j + \sum_{r=1}^{j} (-1)^r Q_{ir} x^{j-r}\ . \tag{3.5}$$

We now seek an expression for the Q_{ir} in terms of the zeros of the auxiliary equation of the fundamental sequence.

4. RECURRENCE RELATION COEFFICIENTS

From (1.3) and (1.4), we see that $\left\{x_n^{(j)}\right\}$ is the product sum of weight j of the quantities $a_{ir}a_{im}$ $(r < m)$. Thus

$$x_n^{(j)} = u_{2n}^{(i)} - \sum_{\lambda>n} a_{i1}^{\lambda_1} a_{i2}^{\lambda_2} \cdots = \sum_{\Sigma\lambda=2n} a_{i1}^{\lambda_1} a_{i2}^{\lambda_2} \cdots - \sum_{\lambda>n} a_{i1}^{\lambda_1} a_{i2}^{\lambda_2} \cdots = \sum_{\Sigma\lambda=2n} \prod_{r=1}^{i} a_{ir}^{\lambda_r}\ . \tag{4.1}$$

For example, when $i = 3$, $j = 3$, and

$$x_1^{(3)} = \Sigma a_{31}a_{32}$$

$$x_2^{(3)} = \Sigma a_{31}^2 a_{32}^2 + \Sigma a_{31}^2 a_{32} a_{33}$$

$$x_3^{(3)} = \Sigma a_{31}^3 a_{32}^3 + \Sigma a_{31}^3 a_{32}^2 a_{32} + \Sigma a_{31}^2 a_{32}^2 a_{33}^2\ .$$

Furthermore, by analogy with (2.4)

(4.2) $$x_n^{(j)} = \sum_{\Sigma r\mu_r = n} (-1)^{n+\Sigma\mu} \frac{(\Sigma\mu)!}{\mu_1!\mu_2!\cdots\mu_n!} \prod_{r=1}^{i} Q_{ir}^{\lambda_r} ,$$

the first few terms of which are

$$x_0^{(j)} = 1$$
$$x_1^{(j)} = Q_{i1}$$
$$x_2^{(j)} = Q_{i1}^2 - Q_{i2}$$
$$x_3^{(j)} = Q_{i1}^3 - 2Q_{i1}Q_{i2} + Q_{i3} ,$$
$$x_4^{(j)} = Q_{i1}^4 - 3Q_{i1}^2Q_{i2} + 2Q_{i1}Q_{i3} + Q_{i2}^2 - Q_{i4} .$$

Write

(4.3) $$\prod_{\substack{r,m=1 \\ r<m}}^{\infty} (1 - a_{ir}a_{im}x) = \sum_{n=0}^{\infty} Q_{in}(-x)^n$$

and then put

(4.4) $$\sum_{n=0}^{\infty} K_{in}x^n = 1 \Big/ \sum_{n=0}^{\infty} Q_{in}(-x)^n .$$

Thus

$$\sum_{n=0}^{\infty} K_{in}x^n = \prod_{r<m} \sum_{n=0}^{\infty} a_{ir}^n a_{im}^n x^n$$

$$= \sum_{n=0}^{\infty} \sum_{\Sigma\lambda=2n} (a_{i1}a_{i2})^{\lambda_{11}}(a_{i1}a_{i3})^{\lambda_{12}} \cdots (a_{i1}a_{ii})^{\lambda_{1,i-1}}(a_{i2}a_{i3})^{\lambda_{21}} \cdots (a_{i2}a_{ii})^{\lambda_{2,i-2}}(a_{i3}a_{i4})^{\lambda_{31}} \cdots x^n$$

$$= \sum_{n=0}^{\infty} \sum_{\Sigma\lambda_r = rn} \prod_{r=1}^{i} a_{ir}^{\lambda_r}x^n ,$$

in which

$$\lambda_r = \sum_{m+v=r} \lambda_{mv} + \sum_{s=1}^{i-r} \lambda_{rs} ,$$

so that

(4.5) $$K_{in} = \sum_{\Sigma\lambda=2n} \prod_{r=1}^{i} a_{ir}^{\lambda_r} .$$

In other words, K_{in} is the product sum of weight n of the quantities $a_{ir}a_{im}$ $(r < m)$, and so $K_{in} = x_n^{(j)}$.

If we write $-x$ for x in (4.4) we get

$$\Sigma Q_{in}x^n = 1/\Sigma K_{in}(-x)^n ,$$

which can also be obtained by leaving x unchanged in (4.4) and simply interchanging the symbols Q and K.

We next expand the right-hand side of (4.4) by the multinomial theorem to obtain

(4.6) $$x_n^{(j)} = K_{in} = \sum_{\Sigma r\mu_r = n} (-1)^{n+\Sigma\mu} \frac{(\Sigma\mu)!}{\mu_1!\mu_2!\cdots\mu_n!} \prod_{r=1}^{n} Q_{ir}^{\mu_r} .$$

An interchange of symbols yields

(4.7) $$Q_{in} = \sum_{\Sigma r\mu_r = n} (-1)^{n+\Sigma\mu} \frac{(\Sigma\mu)!}{\mu_1!\mu_2!\cdots\mu_n!} \prod_{r=1}^{n} \left(x_r^{(j)}\right)^{\mu_r} ,$$

which is an expression for Q_{ir} in terms of a_{ir}, since from (4.5),

$$x_n^{(j)} = \sum_{\Sigma\lambda=2n} \prod_{r=1}^{i} a_{ir}^{\lambda_r},$$

where

$$\lambda_r = \sum_{m+s=r} \lambda_{ms} + \sum_{w=1}^{i-r} \lambda_{rw}.$$

For example,

$$Q_{31} = x_1^{(3)} = \Sigma a_{31}a_{32},$$

$$Q_{32} = -(x_1^{(3)})^2 + x_2^{(3)} = -(\Sigma a_{31}^2 a_{32} + 2\Sigma a_{31}^2 a_{32}a_{33}) + (\Sigma a_{31}^2 a_{32}^2 + \Sigma a_{31}^2 a_{32}a_{33}) = -\Sigma a_{31}^2 a_{32}a_{33}.$$

Q_{in} can also be expressed in terms of $u_n^{(i)}$ from (4.1), and $u_n^{(i)}$ can be expressed in terms of P_{in} in (2.4), so that Q_{in} can be expressed in terms of P_{in} if desired. This has already been illustrated for (2.2).

Another formula for $x_n^{(j)}$ can be given by analogy with (2.5). Since

$$x_n^{(j)} = \sum_{\Sigma\lambda=2n} \prod_{r=1}^{i} a_{ir}^{\lambda_r}$$

and

$$u_n^{(i)} = \sum_{\Sigma\lambda=n} \prod_{r=1}^{i} a_{ir}^{\lambda_r},$$

then

$$x_n^{(j)} = \sum_{r=1}^{i} a_{ir}^{i+2n-1} \Big/ \prod_{r>s} (a_{ir} - a_{is}),$$

which is somewhat surprising since it is expressed entirely in terms of the zeros of $f_i(x)$ rather than $g_j(x)$.

5. JARDEN'S QUERY

Corresponding to the "fundamental" sequence $\{u_n^{(i)}\}$ and by analogy with Lucas' second-order "primordial" sequence [4], we define an i^{th} order primordial sequence by

(5.1)
$$v_n^{(i)} = \sum_{r=1}^{i} P_{ir}v_{n-r}^{(i)} \qquad n > 0,$$
$$v_n^{(i)} = i \qquad n = 0,$$
$$v_n^{(i)} = 0 \qquad n < 0,$$

so that

$$v_n^{(i)} = \sum_{r=1}^{i} a_{ir}^n.$$

Jarden [3, p. 88] suggests that it would be interesting to determine (in our notation)

(5.2)
$$u_{2n}^{(i)} - u_n^{(i)}v_n^{(i)}$$

since

$$u_{2n}^{(2)} - u_n^{(2)}v_n^{(2)}$$

is of great importance in the arithmetic of second-order sequences. We have already seen the auxiliary equation for $\{u_{2n}^{(i)}\}$ in (3.3). Thus

$$u_{2n}^{(i)} - u_n^{(i)}v_n^{(i)} = \sum_{r=1}^{i} A_{ir}a_{ir}^{2n} - \sum_{r=1}^{i} A_{ir}a_{ir}^n \sum_{m=1}^{i} a_{im}^n = -\sum_{\substack{r,s=1\\ r<s}}^{i} (A_{ir}+A_{is})(a_{ir}a_{is})^n = \sum_m C_{jm}\beta_{jm}^n = y_n^{(j)},$$

where $\beta_{jm} = a_{ir}a_{is}$, $r < s$, and $C_{jm} = -(A_{ir}+A_{is})$. Note that since

$$u_0^{(i)} = 1 = \sum_{r=1}^{i} A_{ir}$$

$$v_0^{(j)} = \sum_m C_{jm} = -2\sum_{r=1}^{i} A_{ir} = 2 .$$

Furthermore, the zeros of the auxiliary equations of

$$\left\{x_n^{(j)}\right\} \quad \left\{v_n^{(j)}\right\}$$

are the same, namely β_{jr}. The β_{jr} also come into other properties of recurrence relations such as the quadratic forms of divisors of $v_{2n}^{(2)}$ determined by Lucas [4, p. 43].

The mention of these examples is made to point out that though we have restricted our study of these "*i*-related sequences of order *j*" to expressions for auxiliary equations (3.4) and (3.5) and for recurrence relation coefficients (4.3), (4.5) and (4.7), they can be used in other situations.

6. HORADAM'S PYTHAGOREAN TRIPLES

This basic approach of analyzing and recombining the zeros of the auxiliary equation might be the only fruitful one in studying other properties of recurrence relations of arbitrary order. For instance, Shannon and Horadam [8] proved a general Pythagorean theorem for

$$f_n^{(i)} = \sum_{r=1}^{i} f_{n-r}^{(i)}$$

with suitable initial values. It was shown that

$$(6.1) \qquad (f_n^{(i)} f_{n+i+1}^{(i)})^2 + (2f_{n+i}^{(i)}(f_{n+i}^{(i)} - f_n^{(i)}))^2 = ((f_n^{(i)})^2 + 2f_{n+i}^{(i)} - f_n^{(i)}))^2 ,$$

and that all Pythagorean triples can be formed from such recurrence triples. The case $i = 2$ is the situation studied first by Horadam [2].

The proof of (6.1) cannot be extended to a similar expression for $\left\{u_n^{(i)}\right\}$ because of the presence of the coefficients P_{ir} in the recurrence relation for $\left\{u_n^{(i)}\right\}$. An essential feature of the proof of (6.1) was the result

$$2f_{n+i}^{(i)} - f_{n+i+1}^{(i)} = f_n^{(i)} .$$

This suggests that we consider

$$(6.2) \qquad 2u_{n+i}^{(i)} - u_{n+i+1}^{(i)} = 2\sum_{r=1}^{i} A_{ir}\alpha_{ir}^{n+i} - \sum_{r=1}^{i} A_{ir}\alpha_{ir}^{n+i+1}$$

which follows from (2.3).

The right-hand side of (6.2) becomes

$$\begin{aligned}\sum_{r=1}^{i} A_{ir}\left(2\alpha_{ir}^{n+i} - \alpha_{ir}^{n+i+1}\right) &= \sum_{r=1}^{i} A_{ir} \sum_{s=1}^{i} P_{is}\alpha_{ir}^{n+i-s}(2-\alpha_{ir}) \\ &= \sum_{r=1}^{i}\sum_{s=1}^{i} A_{ir}P_{is}\alpha_{ir}^{n+i-s}(2-\alpha_{ir}) = \sum_{s=1}^{i} P_{is}\sum_{r=1}^{i} A_{ir}(2-\alpha_{ir})\alpha_{ir}^{n+i-s} \\ &= \sum_{s=1}^{i} P_{is}\left(\sum_{r=1}^{i} B_{ir}\alpha_{ir}^{n+i-s}\right) ,\end{aligned}$$

where we have set

$$B_{ir} = A_{ir}(2-\alpha_{ir}).$$

Suppose further that

$$z_n^{(i)} = \sum_{r=1}^{i} B_{ir}\alpha_{ir}^{n+i-s}$$

so that $\{z_n^{(i)}\}$ satisfies the same recurrence relation as $\{u_n^{(i)}\}$ but has different initial conditions (which give rise to the B_{ir}). Then

$$z_n^{(i)} = \sum_{r=1}^{i} P_{ir} z_{n-r}^{(i)} \quad .$$

Proof:

$$z_{n-r}^{(i)} = \sum_{s=1}^{i} B_{is}\alpha_{is}^{n-r} \; .$$

$$\sum_{r=1}^{i} P_{ir} z_{n-r}^{(i)} = \sum_{r=1}^{i} P_{ir} \sum_{s=1}^{i} B_{is}\alpha_{is}^{n-r} = \sum_{r=1}^{i} P_{ir} \sum_{s=1}^{i} (2A_{is}\alpha_{is}^{n-r} - A_{is}\alpha_{is}^{n-r+1}) = 2\sum_{r=1}^{i} P_{ir} u_{n-r}^{(i)} - \sum_{r=1}^{i} P_{ir} u_{n-r+1}^{(i)}$$

$$= 2u_n^{(i)} - u_{n+1}^{(i)} = \sum_{r=1}^{i} 2A_{ir}\alpha_{ir}^n - \sum_{r=1}^{i} A_{ir}\alpha_{ir}^{n+1} = \sum_{r=1}^{i} A_{ir}(2-\alpha_{ir})\alpha_{ir}^n = \sum_{r=1}^{i} B_{ir}\alpha_{ir}^n = z_n^{(i)} \; ,$$

as required. So

$$\sum_{s=1}^{i} P_{is}\left(\sum_{r=1}^{i} B_{ir}\alpha_{ir}^{n+i-s}\right) = \sum_{s=1}^{i} P_{is} z_{n+i-s}^{(i)} = z_{n+i}^{(i)} \; .$$

Thus we have proved

(6.3) $$2u_{n+i}^{(i)} - u_{n+i+1}^{(i)} = z_{n+i}^{(i)} \; ,$$

from which it follows immediately that

$$2u_{n+i}^{(i)} + u_{n+i+1}^{(i)} = 4u_{n+i}^{(i)} - z_{n+i}^{(i)} \; .$$

Thus we have

$$(2u_{n+i}^{(i)} - u_{n+i+1}^{(i)})(2u_{n+i}^{(i)} + u_{n+i+1}^{(i)}) = z_{n+i}^{(i)}(4u_{n+i}^{(i)} - z_{n+i}^{(i)})$$

which becomes

$$4(u_{n+i}^{(i)})^2 - (u_{n+i+1}^{(i)})^2 = z_{n+i}^{(i)}(4u_{n+i}^{(i)} - z_{n+i}^{(i)}) \; .$$

This can be rearranged as

$$(u_{n+i+1}^{(i)})^2 = (z_{n+i}^{(i)})^2 + 4u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}) \; .$$

Multiply each side of this equation by $(z_{n+i}^{(i)})^2$ and

$$(z_{n+i}^{(i)} u_{n+i+1}^{(i)})^2 = (z_{n+i}^{(i)})^4 + 4(z_{n+i}^{(i)})^2 u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}) \; .$$

Add

$$(2u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}))^2$$

to each side to get

(6.4) $$(z_{n+i}^{(i)} u_{n+i+1}^{(i)})^2 + (2u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}))^2 = ((z_{n+i}^{(i)})^2 + 2u_{n+i}^{(i)}(u_{n+i}^{(i)} - z_{n+i}^{(i)}))^2 \; .$$

Equation (6.4) may be considered as an extension of (6.1) and a generalization of Horadam's Pythagorean theorem, since (6.4) reduces to (6.1) when $P_{ir} = 1$ $(r = 1, 2, \cdots, i)$ because $z_{n+i}^{(i)} = u_n^{(i)}$ then (from (6.3) above and Eq. 9 of [7]).

Thus we have shown how three properties of a fundamental recursive sequence of arbitrary order can be generalized by analyzing and recombining the zeros of the auxiliary equation so that the essential features of the properties are revealed.

It is worth noting that Marshall Hall [1] looked at the divisibility properties of a third-order sequence with auxiliary equation roots $\alpha_1^2, \alpha_2^2, \alpha_1\alpha_2$ formed from a second-order sequence with auxiliary equation roots α_1 and α_2.

Thanks are due to Professor A.F. Horadam of the University of New England, New South Wales, for his comments on a draft of this paper.

REFERENCES

1. Marshall Hall, "Divisibility Sequences of Third Order," *Amer. Journal of Math.,* Vol. 58, 1936, pp. 577–584.
2. A.F. Horadam, "Fibonacci Number Triples," *Amer. Math. Monthly,* Vol. 68, 1961, pp. 751–753.
3. Dov Jarden, *Recurring Sequences,* Riveon Lematematika, Jerusalem, 1966.
4. Edouard Lucas, *The Theory of Simply Periodic Numerical Functions* (Douglas Lind, Ed.), Fibonacci Association, California, 1969.
5. Percy A. Macmahon, *Combinatory Analysis,* Vol. 1, University Press, Cambridge, 1915.
6. A.J. van der Poorten, "A Note on Powers of Recurrence Sequences," *Duke Math. Journal,* submitted.
7. A.G. Shannon, "A Fundamental Recursive Sequence Related to a Contraction of Bernoulli's Iteration," *The Fibonacci Quarterly,* submitted.
8. A.G. Shannon and A.F. Horadam, "A Generalized Pythagorean Theorem," *The Fibonacci Quarterly,* Vol. 9, No. 3 (May, 1971), pp. 307–312.

****

LETTER TO THE EDITOR

January 1, 1973

Dear Prof. Hoggatt:

HAPPY NEW YEAR. Here is a problem:

Let $p_1, p_2, \cdots, p_s$ be given primes and let $a_1 < a_2 < \cdots$ be the integers composed of the primes $p_1, p_2, \cdots p_r$. Put

$$A_k = [a_1, a_2, \cdots, a_k]$$

(least common multiple), then

$$\sum_{k=1}^{\infty} \frac{1}{A_k}$$

is irrational. (Conjecture) This is undoubtedly true, but I cannot prove it. All I can show is that

$$\sum_{k=1}^{\prime} \frac{1}{A_k}$$

is irrational, where in Σ' the summation is extended only over the distinct A_k's (i.e., if

$$[a_1, \cdots, a_k] = [a_1, \cdots, a_{k+1}] ,$$

then we count only one of the $1/[a_1, \cdots, a_k]$).

Regards to all,
Paul Erdös