ON HALSEY'S FIBONACCI FUNCTION

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Halsey in [1] defined a Fibonacci function by

$$(1)\qquad F_u = \sum_{k=0}^{m}\left[(u-k)\int_0^1 x^{u-2k-1}(1-x)^k dx\right]^{-1},$$

where m is the integer in the range $(u/2)-1 \leqslant m < (u/2)$.

This definition was criticized by Parker [2] for (a) being restricted to rational u's and (b) destroying the relation

$$(2)\qquad F_{u+1} = F_u + F_{u-1}\,.$$

Neither of these criticizms are quite fair. Firstly, there is nothing in Halsey's paper to prevent (1) from defining F_u for all real u and secondly (2) is still satisfied for approximately half of the real values of u and it is generalized in the other cases. This we show below.

Firstly, we express F_u in the more convenient form given implicitly by Halsey:

$$(3)\qquad F_u = \sum_{k=0}^{m}\binom{u-k-1}{k},$$

where $(u/2)-1 \leqslant m < (u/2)$ and m is an integer.

Now if $(u/2)-½ \leqslant m < (u/2)$, then

$$\frac{u+1}{2} - 1 \leqslant m < \frac{u}{2} < \frac{u+1}{2}$$

so that

$$F_{u+1} = \sum_{k=0}^{m}\binom{u+1-k-1}{k}$$

with the same m.

Also,

$$\frac{u-1}{2} - 1 \leqslant m-1 < \frac{u}{2} - 1 < \frac{u-1}{2}$$

so that

$$F_{u-1} = \sum_{k=0}^{m-1}\binom{u-1-k-1}{k}$$

also with the same m.

Now

$$F_{u+1} - F_u = \sum_{k=1}^{m} \frac{(u-k)!}{(u-2k)!k!} - \frac{(u-k-1)!}{(u-2k-1)!k!} = \sum_{k=1}^{m} \frac{(u-k-1)!}{(u-2k)!(k-1)!}$$

$$= \sum_{q=0}^{m-1} \frac{(u-1-q-1)!}{(u-1-2q-1)!q!}, \text{ where } q = k-1$$

$$= \sum_{q=0}^{m-1} \binom{u-1-q-1}{q} = F_{u-1}.$$

If on the other hand $(u/2) - 1 \leqslant m < (u/2) - \tfrac{1}{2}$, then

$$\frac{u+1}{2} - 1 < \frac{u}{2} < m+1 < \frac{u+1}{2}$$

so that

$$F_{u+1} = \sum_{k=0}^{m+1} \binom{u+1-k-1}{k},$$

where we are still using m as in (3).

Now

$$F_{u+1} - F_u = \binom{u-m-1}{m+1} + \sum_{k=1}^{m} \frac{(u-k)!}{(u-2k)!k!} - \frac{(u-k-1)!}{(u-2k-1)!k!}$$

$$= \binom{u-m-1}{m+1} + \sum_{q=0}^{m-1} \binom{u-1-q-1}{q} \text{ as before}$$

$$= \binom{u-m-1}{m+1} - \binom{u-1-m-1}{m} + F_{u-1} = F_{u-1} + \frac{(u-m-1)!}{(u-2m-2)!(m+1)!} - \frac{(u-m-2)!}{(u-2m-2)!m!}$$

$$= F_{u-1} + \frac{(u-m-2)!}{(u-2m-3)!(m+1)!} = F_{u-1} + \binom{u-m-2}{m+1}.$$

Thus we have for $2m < u \leqslant 2m+1$ that (2) applies and for $2m+1 < u \leqslant 2m+2$

$$F_{u+1} = F_u + F_{u-1} + \binom{u-m-2}{m+1}, \tag{5}$$

where m is an integer.

Equation (5) also reduces to (2) when u is an integer and is also verified by Halsey's tables for F_u.

REFERENCES

1. Eric Halsey, "The Fibonacci Number F_u where u is not an Integer," *The Fibonacci Quarterly,* Vol. 2, No. 2 (April 1965), pp. 147–152.
2. Francis D. Parker, "A Fibonacci Function," *The Fibonacci Quarterly,* Vol. 6, No. 1 (Feb. 1968), pp. 1–2.

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