GENERALIZATIONS OF EULER'S RECURRENCE FORMULA FOR PARTITIONS

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INTRODUCTION

In 1954, H. L. Alder [1] showed that, as a generalization of the Rogers-Ramanujan identities, there exist polynomials $G_{k,n}(x)$ such that

$$(1)\qquad \prod_{\substack{n=1\\ n\not\equiv 0,\pm k \,(\mathrm{mod}\ 2k+1)}}^{\infty} (1-x^n)^{-1} = \sum_{n=0}^{\infty} \frac{G_{k,n}(x)}{(1-x)(1-x^2)\cdots(1-x^n)},$$

and

$$(2)\qquad \prod_{\substack{n=1\\ n\not\equiv 0,\pm 1 \,(\mathrm{mod}\ 2k+1)}}^{\infty} (1-x^n)^{-1} = \sum_{n=0}^{\infty} \frac{G_{k,n}(x)x^n}{(1-x)(1-x^2)\cdots(1-x^n)},$$

where k is a positive integer and the left-hand side of (1) is the generating function for the number of partitions into parts $\not\equiv 0, \pm k \pmod{2k+1}$, while the left-hand side of (2) is the generating function for the number of partitions into parts $\not\equiv 0, \pm 1 \pmod{2k+1}$. As Alder remarks, when $k = 2$, identities (1) and (2) reduce to the Rogers-Ramanujan identities for which $G_{2,n}(x) = x^{n^2}$.

Alder showed that identities similar to (1) and (2) exist for the generating function for the number of partitions into parts $\not\equiv 0, \pm(k-r) \pmod{2k+1}$ for all r with $0 \leqslant r \leqslant k-1$, so that, for a given modulus $2k+1$, there exist k such identities.

We shall show in this paper that a similar generalization is possible for recursion formulae for the number of unrestricted or restricted partitions of n. The best known of these is the Euler identity for the number of unrestricted partitions of n:

$$(3)\qquad p(n) = \sum_{j} (-1)^{j+1} p\left(n - \frac{3j^2 \pm j}{2}\right),$$

where the sum extends over all positive integers j for which the arguments of the partition function are non-negative. Another recursion formula was obtained by Hickerson [2], who showed that $q(n)$, the number of partitions of n into distinct parts, is given by

$$(4)\qquad q(n) = \sum_{j=-\infty}^{\infty} (-1)^j p(n-(3j^2+j)),$$

where the sum extends over all integers j for which the arguments of the partition function are non-negative.

We shall show here that these and other recursion formulas are special cases of the following

Theorem. If we denote the number of partitions of n into parts $\not\equiv 0, \pm(k-r) \pmod{2k+a}$ by $p'(0, k-r, 2k+a; n)$, then for $0 \le r \le k-1$,

$$p'(0, k-r, 2k+a; n) = \sum_j (-1)^j p\left(n - \frac{(2k+a)j^2 + (2r+a)j}{2}\right), \tag{5}$$

where the sum extends over all integers j for which the arguments of the partition function are non-negative.

Proof. Using Jacobi's triple product identity

$$\prod_{n=0}^{\infty} (1-y^{2n+2})(1+y^{2n+1}z)(1+y^{2n+1}z^{-1}) = \sum_{j=-\infty}^{\infty} y^{j^2} z^j .$$

with

$$y = x^{(2k+a)/2}, \qquad z = -x^{(2r+a)/2},$$

we obtain

$$\prod_{n=0}^{\infty} (1-x^{(2k+a)n+(2k+a)})(1-x^{(2k+a)n+k+r+a})(1-x^{(2k+a)n+k-r}) = \sum_{j=-\infty}^{\infty} (-1)^j x^{\frac{(2k+a)j^2+(2r+a)j}{2}} .$$

Dividing both sides by

$$\prod_{s=1}^{\infty} (1-x^s),$$

the left-hand side becomes the generating function for the number of partitions of n into parts $\not\equiv 0, \pm(k-r) \pmod{2k+a}$. Equating coefficients of x^n in the resulting equation yields the theorem.

Corollary 1. For $r = 0$, we obtain the following recursion formula

$$p'(0, k, 2k+a; n) = \sum_j (-1)^j p\left(n - \frac{(2k+a)j^2 + aj}{2}\right), \tag{6}$$

where it shall be understood here and henceforth

$$\sum_j$$

denotes a sum over all integers for which the arguments of the partition function are non-negative.

Corollary 2. If in (6), we let $k = a = 1$, then $p'(0,1,3;n) = 0$ and

$$\sum_j (-1)^j p\left(n - \frac{3j^2+j}{2}\right) = 0$$

or

$$p(n) = \sum_{j \ne 0} (-1)^{j+1} p\left(n - \frac{3j^2+j}{2}\right),$$

which is the Euler identity (3).

Corollary 3. If in (6), we let $k = 2$, $a = 1$, we obtain a recursion formula for $p'(0,2,5;n)$, which by the first Rogers-Ramanujan identity is equal to the number of partitions of n into parts differing by at least 2, or $q_2(n)$. Therefore we have

$$q_2(n) = \sum_j (-1)^j p\left(n - \frac{5j^2+j}{2}\right). \tag{7}$$

Corollary 4. If in (5), we let $r = k - a$, we obtain

$$p'(0, a, 2k+a; n) = \sum_j (-1)^j p\left(n - \frac{(2k+a)j^2 + (2k-a)j}{2}\right) . \tag{8}$$

Corollary 5. If in (8), we let $k = a = 2$, we obtain a recursion formula for $p'(0,2,6;n)$, which is equal to $q(n)$, the number of partitions of n into odd parts, so that we have

$$q(n) = \sum_j (-1)^j p(n - (3j^2 + j)),$$

which is (4).

REFERENCES

1. H. L. Alder, "Generalizations of the Rogers-Ramanujan Identities," *Pacific J. Math.*, 4 (1954), pp. 161–168.
2. Dean R. Hickerson, "Recursion-type Formulae for Partitions into Distinct Parts," *The Fibonacci Quarterly*, Vol. 11, No. 3 (Oct. 1973), pp. 307–311.

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[Continued from P. 336.]

$$\begin{aligned}(-a/b)(b/-a) &= (a/b)(b/a)(-1/b)\\ &= ((-1/a)/(-1/b))(-1/b)\\ &= -1\end{aligned}$$

if and only if

$$(-1/a) \neq (-1/b) = -1.$$

Therefore,

$$(-a/b)(b/-a) = ((-1/-a)/(-1/b)). \tag{2}$$

Also,

$$(a/-b) = (a/b)(a/-1)'$$

and

$$(-b/a) = (b/a)(-1/a).$$

Since $(a/-1) = 1$, therefore

$$\begin{aligned}(a/-b)(-b/a) &= (a/b)(b/a)(-1/a)\\ &= ((-1/a)/(-1/b))(-1/a)\\ &= -1\end{aligned}$$

if and only if

$$(-1/a) \neq (-1/b) = 1.$$

Therefore,

$$(a/-b)(-b/a) = ((-1/a)/(-1/-b)). \tag{3}$$

Finally,

$$(-a/-b) = -(a/b)(a/-1)(-1/b)$$

and

$$(-b/-a) = -(b/a)(b/-1)(-1/a).$$

[Continued on P. 342.]