DISTRIBUTION OF THE FIRST DIGITS OF FIBONACCI NUMBERS

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In a recent paper [1], J. L. Brown, Jr., and R. L. Duncan showed that the sequence $\{\ell n F_n\}$ is uniformly distributed modulo 1 (u.d. mod 1), where ℓn denotes the natural logarithm and F_n is the n^{th} Fibonacci number. In this paper we show that some modifications of these ideas have some interesting consequences concerning the distribution of the first digits of the Fibonacci numbers. This also answers a question raised in Problem H-125.

It has been noticed, and proved in the probabilitic or measure theoretic sense, that the proportion of physical constants whose first significant digit is less than or equal to a given digit a (in base 10), is $\log_{10}(1+a)$. See [2], [4]. We will show that a wide class of sequences, including the Fibonacci numbers, have a natural density satisfying a similar distribution. Hence, roughly speaking, a large percentage of the Fibonacci numbers have a small first digit.

Let b be a given positive integer. All of our numbers will now be written in base b. Let $\{a_n\}$ be a given sequence of positive numbers. For any digit d in base b, let x_d = number of $n \leqslant x$ such that the first digit of a_n is $\leqslant d$. More generally, if

$$a = a_0 b^k + a_1 b^{k-1} + \cdots, \qquad a_0 \neq 0,$$

define

$$a^* = ab^{-k} \quad ;$$

so that $1 \leqslant a^* < b$ and a and a^* have the same digits. Then if λ is any number $1 \leqslant \lambda \leqslant b$, define x_λ = the number of $n \leqslant x$ such that $a_n^* \leqslant \lambda$. Also, let $x_\lambda(k)$ = the number of $n \leqslant x$ such that $b^k \leqslant a_n \leqslant \lambda b^k$. Hence

$$x_\lambda = \sum x_\lambda(k).$$

We will say that a sequence $\{a_n\}$ is logarithmicly distributed (LD) if $x_\lambda \sim x \log \lambda$, where log means $\log_b$. The connection between this type of distribution of first digits and uniform distribution mod 1 is given by:

Theorem 1. $\{a_n\}$ is LD in base b if and only if $\{\log a_n\}$ is u.d. mod 1.

Proof. $1 \leqslant a_n^* \leqslant \lambda$, if and only if $b^k \leqslant a_n \leqslant \lambda b^k$ for some integer k, if and only if $k \leqslant \log a_n \leqslant k + \log \lambda$ for some integer k, if and only if $(\log a_n) \leqslant \log \lambda$, where (m) denotes the fractional part of m. Hence x_λ = number of $n \leqslant x$ such that $(\log a_n) \leqslant \log \lambda$, and so $x_\lambda \sim x \log \lambda$ if and only if $\{\log a_n\}$ is u.d. mod 1.

Corollary 1. $\{a^n\}$ is LD if and only if a is not a rational power of b.

Proof. This follows immediately from the fact that $\{n \log a\}$ is u.d. mod 1 if and only if $\log a$ is irrational [3].

This last result follows from Weyl's theorem that $\{\beta_j\}$ is u.d. mod 1 if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h \beta_j} = 0$$

for all integers $h > 0$ [3].

Using Weyl's theorem and results concerning trigonometric sums, we can show that sequences such as $\{a^{p_n}\}$ and $\{n^n\}$ are LD where p_n denotes the n^{th} prime.

The following results can be proved using Weyl's theorem, but they can also be obtained directly from the definition of x_λ without recourse to any considerations of uniform distribution.

Theorem 2. If $\{a_n\}$ is LD then

(i) $\{ca_n\}$ is LD for all constants $c > 0$,

(ii) $\{a_n^k\}$ is LD for all positive integers k

(iii) $\{1/a_n\}$ is LD

(iv) $\{\beta_n\}$ is LD if $\beta_n \sim a_n$.

Proof. We illustrate the methods used by proving (iii).

Let $S = \{a_n\}$ be LD and let $S' = \{1/a_n\}$. Let x_λ refer to S, x'_λ refer to S', etc. Then

$$b^k \leqslant \frac{1}{a_n} \leqslant \lambda b^k$$

if and only if

$$\frac{1}{\lambda} b^{-k} \leqslant a_n \leqslant b^{-k} ;$$

hence

$$x'_\lambda(k) = x_b(-k-1) - x_{b/\lambda}(-k-1)$$

which implies

$$\begin{aligned} x'_\lambda &= \sum x'_\lambda(k) = \sum x_b(-k-1) - x_{b/\lambda}(-k-1) \\ &= x_b - x_{b/\lambda} \sim x - x \log(b/\lambda) \sim x \log \lambda . \end{aligned}$$

We are now ready to show:

Theorem 3. $\{F_n\}$ is LD.

Proof.

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} .$$

Since

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0,$$

$$F_n \sim \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}} .$$

Now

$$\left(\frac{1+\sqrt{5}}{2}\right)^n$$

is LD by Corollary 1,

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

is LD by Theorem 2-(i), and so F_n is LD by Theorem 2-(iv).

Theorem 3 is easily extended to other recurrence sequences.

It should also be noted that examples can be constructed which show that

$$\{\alpha_n\} \quad \text{and} \quad \{\beta_n\}$$

LD does not imply that any of

$$\{\alpha_n^{1/k}\}, \quad \{\alpha_n\beta_n\}, \quad \text{or} \quad \{\alpha_n+\beta_n\}$$

are LD. It might be interesting to obtain necessary and/or sufficient conditions for these implications to hold.

REFERENCES

1. J. L. Brown, Jr., and R. L. Duncan, "Modulo One Distribution of Certain Fibonacci-Related Sequences," *The Fibonacci Quarterly,* Vol. 10, No. 3 (April 1972), pp. 277–280.
2. R. Burnby and E. Ellentuck, "Finitely Additive Measures and the First Digit Problem," *Fundamenta Mathematicae,* 65, 1969, pp. 33–42.
3. Ivan Niven, "Irrational Numbers," *Carus Mathematical Monograph No. 11,* The Mathematical Association of America, John Wiley and Sons, Inc., New York.
4. R.S. Pinkham, "On the Distribution of First Significant Digits," *Annals of Mathematical Statistics,* 32, 1961, pp. 1223–1230.

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[Continued from P. 333.]

if and only if

$$(-1/a) \neq (-1/b) = -1\,;$$

$$((-1/a)/(-1/-b)) = -1$$

if and only if

$$(-1/a) \neq (-1/b) = 1\,;$$

$$((-1/-a)/(-1/-b)) = -1$$

if and only if

$$(-1/a) = (-1/b) = 1\,.$$

Now stipulate that

$$(a/-1) = (b/-1) = 1\,.$$

Then, by the classic Law of Quadratic Reciprocity,

$$(a/b)(b/a) = ((-1/a)/(-1/b)). \tag{1}$$

But

$$(-a/b) = (a/b)(-1/b)$$

and

$$(b/-a) = (b/a)(b/-1)\,.$$

Since $(b/-1) = 1$, therefore

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