ARITHMETIC SEQUENCES OF HIGHER ORDER

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I

Definition 1. Given a sequence of numbers

(1) $$a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_n \quad \cdots$$

we call first differences of (1) the numbers of the sequence

$$D_0^1 \quad D_1^1 \quad D_2^1 \quad \cdots \quad D_n^1 \quad \cdots$$

with

$$D_n^1 = a_{n+1} - a_n .$$

By recurrence we define the differences of order k of (1) as the first differences of the sequence of differences of order $k - 1$ of (1), namely the numbers of the sequence

(2) $$D_0^k \quad D_1^k \quad D_2^k \quad \cdots \quad D_n^k \quad \cdots$$

with

(3) $$D_n^k = D_{n+1}^{k-1} - D_n^{k-1} .$$

Observe that (3) is also valid for $k = 1$ if we rename $a_n = D_n^0$.

Definition 2. The sequence (1) is arithmetic of order k if the differences of order k are equal, whereas the differences of order $k - 1$ are not equal. It follows that the differences of order higher than k are null.

Proposition 1. Given a sequence (1), if there exists a polynomial $p(x)$ of degree k with leading coefficient c such that $a_n = p(n)$ for $n = 0, 1, 2, \cdots$ then the sequence is arithmetic of order k and the differences of order k are equal to $k!c$.

Proof. Let $p(x) = cx^k + bx^{k-1} + \cdots$ (the terms omitted are always of less degree than those written). Then

$$a_n = cn^k + bn^{k-1} + \cdots$$

hence

$$D_n^1 = a_{n+1} - a_n = c[(n+1)^k - n^k] + b[(n+1)^{k-1} - n^{k-1}] + \cdots = ckn^{k-1} + \cdots$$

therefore, for the first differences we have a polynomial $p_1(x) = kcx^{k-1} + \cdots$ of degree $k - 1$ and leading coefficient kc such that $D_n^1 = p_1(n)$. Repeating the same process k times we come to the conclusion that $D_n^k = p_k(n)$ for a polynomial $p_k(x)$ of degree zero and leading coefficient $k!c$; hence $D_n^k = k!c$ for $n = 0, 1, 2, \cdots$.

EXAMPLE. The sequence

(4) $$0 \quad 1 \quad 2^k \quad 3^k \quad \cdots \quad n^k \quad \cdots$$

for k a positive integer is arithmetic of order k and $D_n^k = k!$.

Proposition 2. For any sequence (1), arithmetic or not, we have

$$D_n^k = \binom{k}{0} a_{n+k} - \binom{k}{1} a_{n+k-1} + \binom{k}{2} a_{n+k-2} + \cdots \pm \binom{k}{k} a_n .$$

The proof is straightforward using induction on k with the help of (3).

In particular for the sequence (4) we have

(5) $$D_n^k = \binom{k}{0}(n+k)^k - \binom{k}{1}(n+k-1)^k + \binom{k}{2}(n+k-2)^k - \cdots \pm \binom{k}{k} n^k,$$

where the coefficient of n^{k-i} $(i = 0, 1, 2, \cdots, k)$ is

$$\binom{k}{0}\binom{k}{i}k^i - \binom{k}{1}\binom{k}{i}(k-1)^i + \binom{k}{2}\binom{k}{i}(k-2)^i - \cdots \mp \binom{k}{k-1}\binom{k}{i}1^i \pm \binom{k}{k}\binom{k}{i}0^i$$

(we assume that $0^i = 0$ for $i = 1, 2, \cdots, k$ and $0^0 = 1$). Hence the coefficient of n^{k-i} $(i = 1, 2, \cdots, k)$ in (5) is

$$\binom{k}{i}\left[\binom{k}{0}k^i - \binom{k}{1}(k-1)^i + \binom{k}{2}(k-2)^i - \cdots \pm \binom{k}{k-1}1^i\right]$$

and the coefficient of n^k

$$\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \cdots \pm \binom{k}{k}.$$

Since we know that $D_n^k = k!$ we have the remarkable equalities:

(i) $$\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \cdots \pm \binom{k}{k} = 0$$

(which is a very well known fact since it is the development of $(1-1)^k$).

(6) (ii) $$\binom{k}{0}k^i - \binom{k}{1}(k-1)^i + \binom{k}{2}(k-2)^i - \cdots \pm \binom{k}{k-1}1^i = 0$$

for $i = 1, 2, \cdots, k-1$

(7) (iii) $$\binom{k}{0}k^k - \binom{k}{1}(k-1)^k + \binom{k}{2}(k-2)^k - \cdots \pm \binom{k}{k-1}1^k = k!$$

A fourth identity can be obtained from (5) with $n = 0$ and (21), namely

$$\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j}(k-1-j)^k = (k-1)!\binom{k}{2},$$

which can also be written in the form

(iv) $$\binom{k}{0}k^{k+1} - \binom{k}{1}(k-1)^{k+1} + \binom{k}{2}(k-2)^{k+1} - \cdots \pm \binom{k}{k-1}1^{k+1} = k!\binom{k+1}{2}.$$

II

Starting with $k+1$ numbers $A_0, A_1, \cdots, A_k$ we form the "generalized" triangle of Pascal

$$\begin{array}{ccccccccc}
 & & & A_0 & & A_1 & & A_2 & \cdots\ A_k \\
 & & A_0 & & A_0+A_1 & & A_1+A_2 & \cdots & A_k \\
 & A_0 & & 2A_0+A_1 & & A_0+2A_1+A_2 & & A_1+2A_2+A_3\ \cdots & A_k \\
A_0 & & 3A_0+A_1 & & 3A_0+3A_1+A_2 & & \cdots & &
\end{array}$$

....

where each number is the sum of the two above. We observe that the coefficient of A_0 in the h^{th} entry of the n^{th} row is $\binom{n-1}{h-1}$; the coefficient of A_1 is $\binom{n-1}{h-2}$ $\cdots$ and the coefficient of A_k is $\binom{n-1}{h-k-1}$. (We set $\binom{n}{j}$ = 0 whenever $j > n$ or $j < 0$.) Therefore the h^{th} entry of the n^{th} row is

(8) $$\binom{n-1}{h-1}A_0 + \binom{n-1}{h-2}A_1 + \cdots + \binom{n-1}{h-k-1}A_k$$

In particular, for the triangle over the $k+1$ differences $a_0, D_0^1, D_0^2, \cdots, D_0^k$ of the sequence (1) assumed to be arithmetic of order k, in view of (3) and taking into account that $D_0^k = D_1^k = \cdots$ we have

$$\begin{array}{ccccccccc} & & & & a_0 & D_0^1 & D_0^2 \cdots D_0^k & & \\ & & & a_0 & a_1 & D_1^1 & D_1^2 \cdots D_1^k & & \\ & & a_0 & S_1^1 & a_2 & D_2^1 & D_2^2 \cdots D_2^k & & \\ & a_0 & S_1^2 & S_2^1 & a_3 & D_3^1 & D_3^2 \cdots D_3^k & & \\ a_0 & S_1^3 & S_2^2 & S_3^1 & a_4 & D_4^1 & D_4^2 \cdots D_4^k & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \end{array}$$

where

$$S_0^1 = a_0 \quad S_i^1 = S_{i-1}^1 + a_i \text{ and } S_n^k = S_{n-1}^k + S_n^{k-1} .$$

Since in this triangle a_n is the $(n+1)^{th}$ entry of the $(n+1)^{th}$ row, we have

(9) $$a_n = \binom{n}{n} a_0 + \binom{n}{n-1} D_0^1 + \binom{n}{n-2} D_0^2 + \cdots + \binom{n}{n-k} D_0^k$$

or, equivalently,

(10) $$a_n = a_0 + \binom{n}{1} D_0^1 + \binom{n}{2} D_0^2 + \cdots + \binom{n}{k} D_0^k .$$

Observe that if the sequence (1) is not arithmetic we still can construct a "generalized" triangle of Pascal starting with an infinity of entries in the first row.

$$a_0 \quad D_0^1 \quad D_0^2 \quad \cdots \quad D_0^n \quad \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and then instead of (10) we would have

$$a_n = a_0 + \binom{n}{1} D_0^1 + \binom{n}{2} D_0^2 + \cdots + \binom{n}{n} D_0^n .$$

Proposition 2. If (1) is an arithmetic sequence of order k, we can find a polynomial $p(x)$ of degree k such that $a_n = p(n)$.

Proof. $$p(x) = a_0 + \binom{x}{1} D_0^1 + \binom{x}{2} D_0^2 + \cdots + \binom{x}{k} D_0^k$$

with

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$$

is obviously a polynomial of degree k and in view of (10), $a_n = p(n)$.

For the partial sum $S_n^1 = a_0 + a_1 + \cdots + a_n$ we have a formula similar to (10). In fact, observing that S_n^1 is the $(n+1)^{th}$ entry of the $(n+2)^{th}$ row in the "generalized" triangle of Pascal, we have

$$S_n^1 = \binom{n+1}{n} a_0 + \binom{n+1}{n-1} D_0^1 + \cdots + \binom{n+1}{n-k} D_0^k$$

or, equivalently,

(11) $$S_n^1 = \binom{n+1}{1} a_0 + \binom{n+1}{2} D_0^1 + \cdots + \binom{n+1}{k+1} D_0^k .$$

Therefore $S_n^1 = q(n)$, where $q(x)$ is a polynomial of degree $k+1$. This was to be expected, since obviously the sequence $S_0^1, S_1^1, \cdots, S_n^1, \cdots$ is arithmetic of order $k+1$.

EXAMPLES. If we apply (11) to the sequences of type (4) with $k = 1, 2, 3, 4$ we obtain the well known formulas

1. $$0+1+2+\cdots+n = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 = \frac{n^2+n}{2}$$

2. $$0+1^2+2^2+\cdots+n^2 = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 + \binom{n+1}{3} 2 = \frac{n(n+1)(2n+1)}{6}$$

3. $$0+1^3+2^3+\cdots+n^3 = \binom{n+1}{1} 0 + \binom{n+1}{2} 1 + \binom{n+1}{3} 6 + \binom{n+1}{4} 6 = \frac{n^4+2n^3+n^2}{4}$$

4. $$0 + 1^4 + 2^4 + \cdots + n^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

III

We now know that the sum

$$S_k(n) = 0 + 1^k + 2^k + \cdots + n^k$$

is given by a polynomial in n of degree $k+1$. The question arises, how to find out the coefficients of this polynomial?

Obviously the coefficient of n^0 is zero, since $S_k(0) = 0$, and the coefficient of n^{k+1} is $1/(k+1)$ as we can see from (11). Hence the polynomial form for $S_k(n)$ is

(12) $$S_k(n) = 1/(k+1)n^{k+1} + h_0 n^k + h_1 n^{k-1} + \cdots + h_{k-1}n$$

for some coefficients $h_0, h_1, \cdots, h_{k-1}$. Since $S_k(n) - S_k(n-1) = n^k$, we have

$$\frac{1}{k+1}[n^{k+1} - (n-1)^{k+1}] + h_0[n^k - (n-1)^k] + h_1[n^{k-1} - (n-1)^{k-1}] + \cdots$$
$$+ h_i[n^{k-i} - (n-1)^{k-i}] + \cdots + h_{k-1} = n^k$$

and taking coefficients of the different powers of n, we have the following equations: (the first is an identity, the rest form a linear system of k equations in k unknowns, which permits to compute recursively $h_0, h_1, \cdots, h_{k+1}$).

(13) $$\begin{cases} \frac{1}{k+1}\binom{k+1}{1} = 1 \\ \frac{1}{k+1}\binom{k+1}{2} - \binom{k}{1}h_0 = 0 \\ \frac{1}{k+1}\binom{k+1}{3} - \binom{k}{2}h_0 + \binom{k-1}{1}h_1 = 0 \\ \cdots\cdots\cdots \\ \frac{1}{k+1}\binom{k+1}{i+1} - \binom{k}{i}h_0 + \binom{k-1}{i-1}h_i - \cdots \pm \binom{k-i+1}{1}h_{i-1} = 0 \\ \cdots\cdots\cdots \\ \frac{1}{k+1}\binom{k+1}{k+1} - \binom{k}{k}h_0 + \binom{k-1}{k-1}h_1 - \cdots \pm \binom{1}{1}h_{k-1} = 0. \end{cases}$$

From the second equation we obtain $h_0 = ½$, independent of k. If we set

(14) $$h_1 = \binom{k}{1}b_1 \qquad h_2 = \binom{k}{2}b_2 \cdots h_{k-1} = \binom{k}{k-1}b_{k-1}$$

and observe that

$$\binom{k-j}{i-j}h_j = \binom{k-j}{i-j}\binom{k}{j}b_j = \binom{k}{i}\binom{i}{j}b_j$$

we can write the i^{th} equation in (13) in the form

$$\frac{1}{i+1}\binom{k}{i} - ½\binom{k}{i} + \binom{k}{i}\binom{i}{1}b_1 - \binom{k}{i}\binom{i}{2}b_2 + \cdots \pm \binom{k}{i}\binom{i}{j}b_j + \cdots \pm \binom{k}{i}\binom{i}{i-1}b_{i-1} = 0$$

or, equivalently:

$$\frac{1}{i+1} - \frac{1}{2} + \binom{i}{1}b_1 - \binom{i}{2}b_2 + \cdots \pm \binom{i}{j}b_j + \cdots \pm \binom{i}{i-1}b_{i-1} = 0.$$

Hence the system (13), after omitting the first two identities reduces to:

(15)
$$\begin{cases} \frac{1}{3} - \frac{1}{2} + \binom{2}{1} b_1 = 0 \\ \frac{1}{4} - \frac{1}{2} + \binom{3}{1} b_1 - \binom{3}{2} b_2 = 0 \\ \cdots\cdots\cdots \\ \frac{1}{i+1} - \frac{1}{2} + \binom{i}{1} b_1 - \binom{i}{2} b_2 + \cdots \pm \binom{i}{j} b_j + \cdots \pm \binom{i}{i-1} b_{i-1} = 0 \\ \cdots\cdots\cdots \\ \frac{1}{k+1} - \frac{1}{2} + \binom{k}{1} b_i - \binom{k}{2} b_2 + \cdots \pm \binom{k}{k-1} b_{k-1} = 0 . \end{cases}$$

We will call Bernoulli numbers the numbers $b_1, b_2, \cdots$. The Bernoulli numbers have over the numbers $h_1, h_2, \cdots$ the advantage that they do not depend on k, as we can see from system (15). Equation (14) permits to calculate for each k the h's in terms of the b's.

Proposition. The even Bernoulli numbers are null.

Proof: Writing $n = 1$ in (12) we have

$$\frac{1}{k+1} + \frac{1}{2} + h_1 + h_2 + \cdots + h_{k-1} = 1 .$$

On the other hand, the last equation in (13) is

$$\frac{1}{k+1} - \frac{1}{2} + h_1 - h_2 + \cdots \pm h_{k-1} = 0.$$

Adding and subtracting these two equations, we obtain:

(16)
$$\begin{cases} h_1 + h_3 + \cdots = \frac{1}{2} - \frac{1}{k+1} \\ h_2 + h_4 + \cdots = 0 \end{cases}$$

The second equation in (16) can be written

$$\binom{k}{2} b_2 + \binom{k}{4} b_4 + \cdots = 0,$$

where the sum is extended to all the subscripts less than or equal to $k - 1$. For $k = 3$ we get $b_2 = 0$; for $k = 5$, $b_4 = 0$, etc., which proves the proposition.

The first equation in (16) for $k = 3, 5, 7, \cdots$ yields the infinite system of equations:

(17)
$$\begin{cases} \binom{3}{1} b_1 = \frac{1}{2} - \frac{1}{4} \\ \binom{5}{1} b_1 + \binom{5}{3} b_3 = \frac{1}{2} - \frac{1}{6} \\ \binom{7}{1} b_1 + \binom{7}{3} b_3 + \binom{7}{5} b_5 = \frac{1}{2} - \frac{1}{8} \\ \cdots\cdots\cdots \end{cases}$$

and for $k = 2, 4, 6, \cdots$ the system

(18)
$$\begin{cases} \binom{2}{1} b_1 = \frac{1}{2} - \frac{1}{3} \\ \binom{4}{1} b_1 + \binom{4}{3} b_3 = \frac{1}{2} - \frac{1}{5} \\ \binom{6}{1} b_1 + \binom{6}{3} b_3 + \binom{6}{5} b_5 = \frac{1}{2} - \frac{1}{7} \\ \cdots\cdots\cdots \end{cases}$$

Subtracting the equations in (18) from those in (17), we have

$$(19)\qquad \begin{cases} \binom{2}{0} b_1 = \dfrac{1}{3\cdot 4} \\ \binom{4}{0} b_1 + \binom{4}{2} b_3 = \dfrac{1}{5\cdot 6} \\ \binom{6}{0} b_1 + \binom{6}{2} b_3 + \binom{6}{4} b_5 = \dfrac{1}{7\cdot 8} \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Any of the infinite systems (17), (18) or (19) permits to find recursively the Bernoulli numbers with odd subscripts.

Substituting in (12) the Bernoulli numbers, we express

$$S_k(n) = 0 + 1^k + \cdots + n^k$$

in the form

$$(20)\qquad S_k(n) = \frac{1}{k+1} n^{k+1} + \frac{1}{2} n^k + b_1 \binom{k}{1} n^{k-1} + b_3 \binom{k}{3} n^{k-3} + \cdots ,$$

where the coefficients of the different powers of n are products of a combinatorial number of k and a number which does not depend on k.

NOTE. If we compute the coefficient of the k^{th} power of n in (11) we have

$$-\frac{(k+1)(k-2)}{2(k+1)!} D_0^k + \frac{1}{k!} D_0^{k-1} .$$

On the other hand for the sequence $0, 1^k, 2^k, \cdots$ that coefficient is ½, and $D_0^k = k!$. Hence, for this particular sequence we have

$$(21)\qquad 2D_0^{k-1} = (k-1)k! .$$

EXAMPLES. From (10) we obtain:

$$b_1 = \frac{1}{12} \qquad b_3 = -\frac{1}{120} \qquad b_5 = \frac{1}{252} \qquad b_7 = -\frac{1}{240} \qquad b_9 = \frac{1}{132}$$

which, substituted in (20) for $k = 1, 2, \cdots, 11$ yields the formulas:

$$1 + 2 + \cdots + n = \frac{1}{2} n^2 + \frac{1}{2} n$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2$$

$$1^4 + 2^4 + \cdots + n^4 = \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{3} n^3 - \frac{1}{30} n$$

$$1^5 + 2^5 + \cdots + n^5 = \frac{1}{6} n^6 + \frac{1}{2} n^5 + \frac{5}{12} n^4 - \frac{1}{12} n^2$$

$$1^6 + 2^6 + \cdots + n^6 = \frac{1}{7} n^7 + \frac{1}{2} n^6 + \frac{1}{2} n^5 - \frac{1}{6} n^3 + \frac{1}{42} n$$

$$1^7 + 2^7 + \cdots + n^7 = \frac{1}{8} n^8 + \frac{1}{2} n^7 + \frac{7}{12} n^6 - \frac{7}{24} n^4 + \frac{1}{12} n^2$$

$$1^8 + 2^8 + \cdots + n^8 = \frac{1}{9} n^9 + \frac{1}{2} n^8 + \frac{2}{3} n^7 - \frac{7}{15} n^5 + \frac{2}{9} n^3 - \frac{1}{30} n$$

$$1^9 + 2^9 + \cdots + n^9 = \frac{1}{10} n^{10} + \frac{1}{2} n^9 + \frac{3}{4} n^8 - \frac{7}{10} n^6 + \frac{1}{2} n^4 - \frac{3}{20} n^2$$

$$1^{10} + 2^{10} + \cdots + n^{10} = \frac{1}{11} n^{11} + \frac{1}{2} n^{10} + \frac{5}{6} n^9 - n^7 + n^5 - \frac{1}{2} n^3 + \frac{5}{66} n$$

$$1^{11} + 2^{11} + \cdots + n^{11} = \frac{1}{12} n^{12} + \frac{1}{2} n^{11} + \frac{11}{12} n^{10} - \frac{11}{8} n^8 + \frac{11}{6} n^6 - \frac{11}{8} n^4 + \frac{5}{12} n^2$$

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