COMBINATORIAL NUMBERS IN $\mathbb{C}^n$

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1. INTRODUCTION

The use of linear algebra in combinatorial number theory was introduced in [4]. The present paper extends the notations and studies the general properties of product functions, i.e., combinatorial number systems in $\mathbb{C}^n$. Among the examples given are n-dimensional Bernoulli and Euler numbers which are useful in the expansion in series of functions in n variables. The methods and notations introduced here will be used in the study of functions and series in $\mathbb{C}^n$ that will be the subject of future investigations.

2. NOTATION

Let I be the set of positive integers, J the set of non-negative integers, and given $n \in I$, let $I(n) \subset I$, and $J(n) \subset J$ be such that if $k \in J(n)$, then $k \leq n$.

In order to avoid confusion we shall write Id for the identity operator or the identity matrix.

For $n \in I$, $k \in I(n)$, $X = [x_1, x_2, \cdots, x_n]$ is an n-dimensional vector and x_k are complex numbers, i.e., $x_k \in \mathbb{C}$, so that $X \in \mathbb{C}^n$.

Let

$$P = [p_1, p_2, \cdots, p_n], \qquad Q = [q_1, q_2, \cdots, q_n],$$

then $W(n) \subset \mathbb{C}^n$ be such that for $P \in W(n)$, $m \in I(n)$, $p_m \in J$, and for $P, Q \in W(n)$, $P \leq Q$, iff for all $m \in I(n)$, $p_m \leq q_m$.

We consider the following special vectors:

(2.1) $$U \in W(n), \quad \text{such that} \quad u_m = 1 \quad \text{for all} \quad m \in I(n),$$

(2.2) $$U(s) \in W(n), \quad \text{such that} \quad u_m = \delta_m^s, \quad \text{for all} \quad m \in I(n),$$

where δ_m^s is the Kronecker delta. It follows that

$$U = \sum_{s=1}^{n} U(s).$$

(2.3) $$Z(s) \in W(n), \quad \text{such that} \quad Z(s) = U - U(s), \quad \text{i.e.,} \quad z_m = 1 - \delta_m^s .$$

(2.4) $$Z(X,s) \in \mathbb{C}^n, \quad \text{such that} \quad z_m = x_m(1 - \delta_m^s), \quad \text{i.e.,} \quad z_s = 0, \quad \text{thus} \quad Z(U,s) = Z(s).$$

We next introduce for $X \in \mathbb{C}^n$

(2.5) $$|X| = \sum_{m=1}^{n} x_m ,$$

so that $|U| = n$, $|U(s)| = 1$, $|Z(s)| = n - 1$, and $|Z(X,s)| = |X| - x_s$.

We finally introduce the inner product in the usual way: If $X, Y \in \mathbb{C}^n$, then

(2.6) $$X \cdot Y = \sum_{m=1}^{n} x_m \bar{y}_m ,$$

where $\bar{y}_m$ is the complex conjugate of y_m. It follows that

(2.7) $$\|X\| = (X \cdot X)^{1/2} = \left(\sum_{m=1}^{n} |x_m|^2 \right)^{1/2} .$$

If, however, $X, Y \in \mathcal{R}^n \subset \mathcal{C}^n$, where $\mathcal{R}^n$ is the space of real n vectors, then

(2.8) $$X \cdot Y = \sum_{m=1}^{n} x_m y_m ,$$

and

(2.9) $$\|X\| = (X \cdot X)^{1/2} = \left[\sum_{m=1}^{n} x_m^2 \right]^{1/2}$$

3. FUNCTIONS OVER $\mathcal{C}^n$

We consider functions $\Phi : \mathcal{C}^n \to \mathcal{C}$.

A monomial in X can be written:

(3.1) $$X^K = \prod_{m=1}^{n} x_m^{k_m} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} ,$$

where $X \in \mathcal{C}^n$, $K \in W(n)$. In particular,

(3.2) $$X^U = \prod_{m=1}^{n} x_m = x_1 x_2 \cdots x_n .$$

A polynomial in X, i.e., a polynomial in n variables, can be written

(3.3) $$f(X,P) = \sum_{K=0}^{P} a(K) X^K ,$$

where the summation is extended over all K such that $K \leqslant P$, $K,P \in W(n)$ and $a(K)$ are numbers. In the generally adopted polynomial sense the degree of $f(X,P)$ is clearly $p = |P|$.

More generally if $\varphi_k(x_k)$, $k \in I(n)$, is a sequence of functions, $\varphi_k : \mathcal{C} \to \mathcal{C}$, then with

$$\Phi = [\varphi_1(x_1), \varphi_2(x_2), \cdots, \varphi_n(x_n)],$$

(3.4) $$\Phi^U = \prod_{k=1}^{n} \varphi_k(x_k) = \varphi(X),$$

is called a *product function* of the functions φ_k.

We study the following examples:

(i) If $\varphi_k = m_k$, $M = [m_1, m_2, \cdots, m_n] \in W(n)$, then with $k \in I(n)$

(3.5) $$\Phi^U = M! = \prod_{k=1}^{n} m_k! ,$$

(ii) If $M \in \mathcal{C}^n$ but $M \notin W(n)$, then we replace factorials by gamma functions thus if $\varphi_k = \Gamma(m_k + 1)$, then

(3.6) $$\Phi^U = \Gamma(M+U) = \prod_{k=1}^{n} \Gamma(m_k + 1).$$

(iii) For $N,M \in W(n)$, $M \leqslant N$, and $k \in I(n)$, we have for

$$\varphi_k = \binom{n_k}{m_k} ,$$

(3.7) $$\Phi^U = \prod_{k=1}^{n} \binom{n_k}{m_k} = \prod_{k=1}^{n} n_k!/m_k!(n_k - m_k)! = N!/M!(N-M)! = \binom{N}{M}.$$

It should be noted that $\binom{N}{M}$ is product function for binomial coefficients and *not* a multinomial coefficient. The corresponding multinomial coefficient would be (cf. [3])

$$\binom{|N|}{[M, N-M]} = \left[\sum_{k=1}^{n} n_k\right]!/M!(N-M)!,$$

where

$$[M, N-M] = [m_1, m_2, \cdots, m_n, n_1 - m_1, n_2 - m_2, \cdots, n_n - m_m] \in W(2n)$$

and clearly $|M| + |N - M| = |N|$.

(iv) For $N, M \in W(n)$, and $A, B \in \mathbb{C}^n$

$$(A+B)^N = \prod_{k=1}^{n} (a_k + b_k)^{n_k} = \prod_{k=1}^{n} \left[\sum_{m_k=0}^{n_k} \binom{n_k}{m_k} a_k^{m_k} b_k^{n_k - m_k}\right],$$

and by regrouping the terms we obtain

(3.8) $$(A+B)^N = \sum_{M=0}^{N} \binom{N}{M} A^M B^{N-M}.$$

(v) For $X \in \mathbb{C}^n$, and with $eU = [e, e, \cdots, e]$, we define

(3.9) $$e^X = (eU)^X = e^{|X|} = \prod_{k=1}^{n} e^{x_k} = \prod_{k=1}^{n} \left[\sum_{m_k=0} x_k^{m_k}/m_k!\right] = \sum_{M=0} X^M/M!,$$

and

$$e^{-X} = \sum_{M=0} (-1)^M X^M/M!,$$

where $(-1)^M = (-1)^{|M|}$.

It will be noted that whenever a summation goes to infinity the upper limit is left out.

4. UMBRAL CALCULUS

Umbral calculus consists in substituting indices for exponents. In [2] the following notation is used for the one dimensional case.

(4.1) $$e^{ax} = \sum_{k=0} x^k a^k/k! \to [\exp ax, a^k = a_k] = \sum_{k=0} x^k a_k/k!$$

(4.2) $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} \to [(a+b)^n, a^k = a_k, b^k = b_k] = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

We shall change this notation and extend it to the n-dimensional case. The umbral expression corresponding to a vector exponent is clearly

(4.3) $$A^K = \prod_{m=1}^{n} a_m^{k_m} \to A(K) = \prod_{m=1}^{n} a_m(k_m),$$

where instead of indices we write variables.

We now introduce the following convention: Whenever an element is to be written umbrally it will be underlined, thus

$$(4.4)\qquad (\underline{a}+\underline{b})^n = \sum_{m=0}^{n} \binom{n}{k} a_k b_{n-k} = \sum_{m=0}^{n} \binom{n}{m} a(k)b(n-k)$$

and with $N, K \in W(n)$

$$(4.5)\qquad (\underline{A}+\underline{B})^N = \prod_{m=1}^{n} [\underline{a}(m)+\underline{b}(m)]^{n_m} = \sum_{K=0}^{N} \binom{N}{K} A(K)B(N-K),$$

but

$$(4.6)\qquad (A+\underline{B})^N = \sum_{K=0}^{N} \binom{N}{K} A^K B(N-K) = \sum_{K=0}^{N} \binom{N}{K} A^{N-K} B(K),$$

and in particular

$$(4.7)\qquad (U+\underline{B})^N = \sum_{K=0}^{N} \binom{N}{K} B(N-K) = \sum_{K=0}^{N} \binom{N}{K} B(K).$$

Similarly for the generalized exponential we have

$$(4.8)\qquad e^{\underline{A}X} = \sum_{K=0} X^K A(K)/K!\,.$$

It should be noted that the last umbral expression (4.8) is the exponential kind generating function for the numbers $A(K)$.

It should be noted that

$$e^{\underline{A}X}e^{\underline{B}X} = \left[\sum_{S=0} X^S A(S)/S!\right]\left[\sum_{T=0} X^T B(T)/T!\right] = \sum_{S=0}\sum_{T=0} X^{S+T} A(S)B(T)/S!T!\,.$$

Let $S+T=K$; observing that $\binom{K}{S} = K!/S!(K-S)!$, we have

$$e^{\underline{A}X}e^{\underline{B}X} = \sum_{K=0}\sum_{S=0} X^K A(S)B(K-S)/S!(K-S)! = \sum_{K=0} (X^K/K!) \sum_{S=0} \binom{K}{S} A(S)B(K-S)\quad ,$$

but according to (4.5) the last sum is equal to $(\underline{A}+\underline{B})^K$, where the binomial coefficients for $S \geqslant K$ are all equal to zero. It follows that

$$(4.9)\qquad e^{\underline{A}X}e^{\underline{B}X} = \sum_{K=0} X^K(\underline{A}+\underline{B})^K/K! = e^{(\underline{A}+\underline{B})X}\quad ,$$

i.e., the symbolic exponential follows the same law of addition as the ordinary exponential.

5. GENERATING FUNCTIONS

Let

$$\Phi(k) = [\varphi(k,1), \varphi(k,2), \cdots, \varphi(k,n)]$$

and using the notation of Section 2, we consider the product function

$$(5.1)\qquad \varphi(k) = [\Phi(k)]^U = \prod_{m=1}^{n} \varphi(k,m).$$

Let $v(t,m)$ be the generating function for the functions $\varphi(k,m)$, i.e.,

$$G\varphi(k,m) = \sum_{k=0} \varphi(k,m)t^k = v(t,m), \tag{5.2}$$

where $m \in I(n)$.

By taking the product

$$\prod_{m=1}^{n} \left[\sum_{k=0} \varphi(k,m)t_m^k \right] = \prod_{m=1}^{n} v(m,t_m) = [V(T)]^U = \Omega(T), \tag{5.3}$$

where $T = [t_1, t_2, \cdots, t_n] \in \mathcal{C}$, and

$$V(T) = [v(1,t_1), v(2,t_2), \cdots, v(n,t_n)] = \mathcal{C}^n$$

we thus obtain the generating function of the product function.

If $\varphi(k,m) = \varphi(k,x_m)$, then $v(m,t_m) = v(x_m, t_m)$ and (5.3) becomes

$$\prod_{m=1}^{n} v(x_m, t_m) = [V(X,T)]^U = \Omega(X,T). \tag{5.4}$$

We can state this result as follows:

PROPOSITION 1. The generating function of the product function of a set of functions is equal to the product of the generating functions of the set of functions.

6. INVERSION OF SERIES

Consider the series

$$A(N) = \sum_{K=0} f(N,K)B(K), \tag{6.1}$$

where the coefficients $f(N,K)$ are known. We say that (6.1) has an inverse if there exists a set of coefficients $g(N,K)$ such that

$$B(N) = \sum_{K=0} g(N,K)A(K), \tag{6.2}$$

both series being convergent.

PROPOSITION 2. If both series (6.1) and (6.2) are absolutely convergent they are inverses of each other if and only if f and g are quasi-orthogonal in the sense of [4] and [5].

PROOF:

$$A(N) = \sum_{K=0} f(N,K)B(K) = \sum_{K=0} f(N,K) \sum_{S=0} g(K,S)A(S) = \sum_{K=0}\sum_{S=0} f(N,K)g(K,S)A(S).$$

Since the series are absolutely convergent, their order can be deranged and the order of summation can be changed, thus

$$A(N) = \sum_{S=0} A(S) \left[\sum_{K=0} f(N,K)g(K,S) \right] = \sum_{S=0} A(S)\delta_N^S,$$

where δ_N^S is the Kronecker-Delta. It follows that

$$\sum_{K=0} f(N,K)g(K,S) = \delta_N^S,$$

which expresses quasi-orthogonality in the sense of [4] and [5].

PROPOSITION 3. $A(N) = (\underline{C} + \underline{B})^N$ and $B(N) = (\underline{G} + \underline{A})^N$ will be inverses of each other if $(\underline{C} + \underline{G})^T = \delta_0^T$.

PROOF. Since

$$A(N) = (\underline{C} + \underline{B})^N = \sum_{K=0}^{N} \binom{N}{K} C(K)B(N-K) = \sum_{K=0}^{N} \binom{N}{K} B(K)C(N-K),$$

$$B(N) = (\underline{G} + \underline{A})^N = \sum_{K=0} \binom{N}{K} G(K)A(N-K) = \sum_{K=0} \binom{N}{K} A(K)G(N-K),$$

where both series involved are finite, i.e., present no problem of convergence, we apply the results of Proposition 2

$$\sum_{K=S}^{N} \binom{N}{K} C(N-K) \binom{K}{S} G(K-S) = \sum_{K=S}^{N} [N/K!(N-K)!][K!/S!(K-S)!]C(N-K)G(K-S)$$

$$= \binom{N}{S} \sum_{K=S}^{N} \binom{N-S}{N-K} C(N-K)G(K-S) = \delta_N^S.$$

Let $K - S = M$, i.e., $N - K = N - S - M$, so that

$$\binom{N-S}{N-K} = \binom{N-S}{N-S-N+K} = \binom{N-S}{M}.$$

The preceding quasi-orthogonality condition can thus be written

$$\binom{N}{S} \sum_{M=0}^{N} \binom{N-S}{M} G(M)C(N-S-M) = \binom{N}{S} (\underline{G} + \underline{C})^{N-S} = \delta_{N-S}^0,$$

or taking $N - S = T$,

$$(\underline{G} + \underline{C})^T = \delta_0^T. \tag{6.3}$$

It will be observed that (6.3) can be written for an arbitrary vector X in either form

$$e^{X(\underline{G}+\underline{C})} = 1, \tag{6.4}$$

or

$$e^{X\underline{G}} = 1/e^{X\underline{C}}. \tag{6.5}$$

7. OPERATORS IN $\mathcal{C}^n$

Let $D(m) = \partial/\partial x_m$, $m \in I(n)$, and $D = [D(1), D(2), \cdots, D(n)]$. We consider the *product operator*

$$\not{D} = D^U = \prod_{m=1}^{n} D(m) \tag{7.1}$$

and more generally $K = [k_1, k_2, \cdots, k_n] \in W(n)$

$$D^K = \prod_{m=1}^{n} [D(m)]^{k_m}. \tag{7.2}$$

Using this notation the n-dimensional Laplace operator can be written

$$\Delta_2 = \sum_{m=1}^{n} \partial^2/\partial x_m^2 = \sum_{m=1}^{n} D^{2U(m)}. \tag{7.3}$$

It is easily seen that for $k \in I(n)$

(7.4) $$C(k) = Z(X,k) + C_k U(k), \qquad c(k) = [C(k)]^U ,$$

$c(k)$ is such that $\not{D}c(k) = 0$. Considering now the vector

(7.5) $$C = [c(1), c(2), \cdots, c(n)]$$

it follows that $\not{D}C = 0$, and, if $\eta(X)$ is a function such that $\eta(0) = 0$, then $\eta(C)$ is the most general expression such that

(7.6) $$\not{D}\eta = 0,$$

where $\eta = \eta(C)$.

Similarly for difference operators we define $E(m)$ such that $E(m)f(x_k) = f(x_k + 1)\delta_m^k$, and

(7.7) $$E = [E(1), E(2), \cdots, E(n)]$$

(7.8) $$\not{E} = E^U = \prod_{m=1}^{n} E(m) .$$

We clearly have $E(m)\varphi(X) = \varphi[X + U(m)]$, and

(7.9) $$\not{E}X = E^U X = X + U, \qquad \not{E}\varphi(X) = \varphi(X+U).$$

The operator $\delta = \not{E} - Id$ is *not* a product operator of the form

$$\prod_{m=1}^{n} [E(m) - Id] .$$

We have however

(7.10) $$\delta X = U, \qquad \delta\varphi(X) = \varphi(X+U) - \varphi(X) .$$

The operator $\Delta(m) = E(m) - Id$ leads clearly to the

(7.11) $$\Delta = [\Delta(m_1), \Delta(m_2), \cdots, \Delta(m_n)]$$

(7.12) $$\not{\Delta} = \Delta^U = \prod_{k=1}^{n} \Delta(m_k).$$

It follows that $\not{\Delta}X = U$, but the general expression of $\not{\Delta}X^K$ is rather complicated.

The operator $M(m) = [E(m) + Id]/2$ leads similarly to

(7.13) $$M = [M(m_1), M(m_2), \cdots, M(m_n)]$$

and

(7.14) $$\not{M} = M^U = \prod_{k=1}^{n} M(m_k).$$

A more systematic study of the operators introduced here as well as the corresponding functional equations will be published in the future. We introduce here only what we need in view of the applications given.

8. RECURRENCE RELATIONS AND FUNCTIONAL EQUATIONS

Let $m \in I(n)$, and $a(m)$ be a one dimensional sequence of numbers satisfying a recurrence relation of the form

(8.1) $$\sum_{m=0}^{p} b(p,m)a(m) = 0, \qquad p \in J .$$

Let $k \in I(n)$, $m_k \in J$, $M = [m_1, m_2, \cdots, m_n]$, and

$$A(M) = [a(m_1), a(m_2), \cdots, a(m_n)]$$

and the associated product function be

(8.2) $$a(M) = [A(M)]^U = \prod_{k=1}^{n} a(m_k) .$$

By writing the product for (8.1) we obtain

$$\prod_{k=1}^{n}\left[\sum_{m_k=0}^{p} b(p,m_k)a(m_k)\right] = 0\,.$$

Regrouping the terms we obtain

$$\sum_{M=0}^{pU} b(p,M)a(M) = 0, \tag{8.3}$$

where $B(p,M) = [b(p,m_1), b(p,m_2), \cdots, b(p,m_n)]$, and

$$b(p,M) = [B(p,M)]^U = \prod_{k=1}^{n} b(p,m_k).$$

Clearly $pU = [p, p, \cdots, p]$. We can state this result as follows.

PROPOSITION 4. If a sequence of numbers $a(m)$ satisfies a recurrence relation of the form (8.1) then the product function of the numbers $a(m)$, i.e., $a(M)$ satisfies a recurrence relation of the form (8.3).

If $\omega(m)$ is an operator such that

$$\omega(m)f(x_k) = \varphi(x_k)\delta_m^k\,, \tag{8.4}$$

where δ_m^k is the Kronecker delta.

Let $X \in \mathcal{C}$,

$$F(X) = [f(1,x_1), f(2,x_2), \cdots, f(n,x_n)] \in \mathcal{C}^n, \quad \Phi(X) = [\varphi(1,x_1), \varphi(2,x_2), \cdots, \varphi(n,x_\chi)] \in \mathcal{C}^n\,,$$

$$\Omega = [\omega(1), \omega(2), \cdots, \omega(n)] \quad \text{and} \quad f(X) = [\Phi(X)]^U, \quad \varphi(X) = [\Phi(X)]^U, \quad \omega = \Omega^U,$$

then

$$\omega f(X) = \varphi(X). \tag{8.4}$$

9. EXAMPLES

(i) Consider the numbers $a_m = a(m)$ defined in [1] p. 231. They satisfy the relation

$$\sum_{m=0}^{n-1} a(m)(n-m)! = 0. \tag{9.1}$$

These numbers are the coefficients of the Bernoulli polynomials

$$\varphi_n(x) = \varphi(n,x) = \sum_{m=0}^{n} a(m)x^{n-m}/(n-m)!\,. \tag{9.2}$$

The numbers

$$B_m = B(m) = m!a(m) \tag{9.3}$$

are called Bernoulli numbers and satisfy the relation

$$(1+\underline{B})^n - B(n) = 0. \tag{9.4}$$

By using the Bernoulli numbers the polynomials of (9.2) can be written

$$\varphi(n,x) = (x+\underline{B})^n/n!\,. \tag{9.2a}$$

We introduce $M = [m_1, m_2, \cdots, m_n] \in W(n)$ and $A(M) = [a(m_1), a(m_2), \cdots, a(m_n)]$,

$$a(m) = [A(M)]^U = \prod_{k=1}^{n} a(m_k), \tag{9.5}$$

as well as

$$B(M) = [B(m_1), B(m_2), \cdots, B(m_n)],$$

(9.6) $$B(n,M) = [B(M)]^U = \prod_{k=1}^{n} B(m_k).$$

The numbers $B(n,M)$ are called the n-dimensional Bernoulli numbers. According to Section 8 we clearly have

(9.7) $$\sum_{M=0}^{P} a(M)/(P-M)! - a(P) = 0$$

and

(9.8) $$[U + \underline{B}(n)]^P - B(n,P) = 0.$$

(9.7) and (9.8) are the recurrence relations for the $a(M)$ and the n-dimensional Bernoulli numbers.

Consider next

$$P = [p_1, p_2, \cdots, p_n] \in W(n), \quad \Phi(P) = [\varphi(p_1, x_1), \varphi(p_2, x_2), \cdots, \varphi(p_n, x_n)]$$

and

(9.9) $$\varphi(P,X) = \prod_{k=1}^{n} \varphi(p_k, x_k) = \sum_{K=0}^{P} a(K)X^{P-K}/(P-K)! = \sum_{K=0}^{P} B(n,K)X^{P-K}/K!(P-K)! = [X + \underline{B}(n)]^P ,$$

from where it is easily seen that (cf. (7.1))

(9.10) $$\mathcal{D}\varphi(P,X) = \varphi(P-U, X).$$

On the other hand, according to [1], p. 231,

$$\Delta(k)\varphi(p_k, x_k) = x_k^{p_k-1}/(p_k-1)!,$$

so that by multiplication over k we obtain

(9.11) $$\Delta\varphi(P,X) = X^{P-U}/(P-U)!.$$

According to Section 5 and [1], we obtain the generating function of the n-dimensional Bernoulli numbers as follows:

(9.12) $$\Omega(T) = T^U/(e^T - 1) = \sum_{M=0} B(n,M)T^M .$$

(ii) Consider the numbers $e(m)$ defined by the recurrence relation (cf. [1], p. 289)

(9.13) $$e(n) + \sum_{k=0}^{n} e(k)/(n-k)! = 0.$$

The numbers $e(m)$ are the coefficients of the Euler polynomials

(9.14) $$\eta(n,x) = \sum_{k=0}^{n} e(k)x^{n-k}/(n-k)!.$$

The numbers $t(n) = 2^n e(n)n!$ are called the tangent coefficients (cf. [1], p. 298) and the numbers

(9.15) $$\epsilon(n) = (1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t(k)$$

Euler numbers. According to [1], the tangent coefficients satisfy the recurrence relation

(9.16) $$(2+\underline{t})^n + t(n) = 0.$$

It is shown in [1] that (9.15) can be inverted to give $t(n) = [\underline{\epsilon} - 1]^n$. It follows that

(9.17) $$[\underline{\epsilon}+1]^n + [\underline{\epsilon}-1]^n = 0, \quad n > 0.$$

As before we introduce $M \in W(n)$ and $\eta(M) = [e(m_1), e(m_2), \cdots, e(m_n)]$, with

$$\eta(n,M) = [\eta(M)]^U = \prod_{m=1}^{n} e(m_k).$$

The n-dimensional tangent coefficients will be $T(M) = [t(m_1), t(m_2), \cdots, t(m_n)]$, so that

$$t(n,M) = [T(M)]^U = \prod_{k=1}^{n} t(m_k).$$

Finally let $\epsilon(M) = [\epsilon(m_1), \epsilon(m_2), \cdots, \epsilon(m_n)]$, so that

$$\epsilon(n,M) = [\epsilon(M)]^M = \prod_{k=1}^{n} \epsilon(m_k),$$

where the numbers $\epsilon(n,M)$ are called the n-dimensional Euler numbers. It is easily seen, like in the case of the Bernoulli numbers, that

(9.18) $$[\underline{\epsilon}(n)+1]^P + [\underline{\epsilon}(n)-1]^P = 0, \quad P > 0,$$

(9.19) $$t(n,P) + [2U + \underline{T}(n)]^P = 0, \quad P > 0,$$

(9.20) $$t(n,K) = K!2^K e(n,K),$$

(9.21) $$\epsilon(n,P) = [U + \underline{T}(n)]^P,$$

(9.22) $$t(n,M) = [\epsilon(n) - U]^M.$$

We introduce in the same way the n-dimensional Euler polynomials: Let

$$H(P) = [\eta(p_1, x_1), \eta(p_2, x_2), \cdots, \eta(p_n, x_n)],$$

where $P \in W(n)$. It follows that

(9.23) $$\eta(P,K) = \prod_{k=1}^{n} \eta(p_k, x_k) = \sum_{K=0}^{P} e(n,K)X^{P-K}/(P-K)!,$$

which defines the n-dimensional Euler polynomials.

It can easily be checked that similarly to the one-dimensional case we have

(9.24) $$\mathcal{D}\eta(P,X) = \eta(P-U, X)$$

and

(9.25) $$\mathcal{M}\eta(P,X) = X^P/P!.$$

According to Section 8 we obtain the following generating function for the Euler numbers $\epsilon(n,K)$ and the numbers $e(n,K)$

(9.26) $$G\epsilon(n,P) = 2/[e^T + e^{-T}] = \sum_{K=0} \epsilon(n,K)T^K/K!$$

(9.27) $$Ge(n,P) = 2/[e^T + 1] = \sum_{K=0} e(n,K)T^K.$$

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