CONSECUTIVE INTEGER PAIRS OF POWERFUL NUMBERS AND RELATED DIOPHANTINE EQUATIONS

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1. INTRODUCTION

S. W. Golomb [1] defined a *powerful number* to be a positive integer r, such that p^2 divides r whenever the prime p divides r, and discussed consecutive integer pairs of powerful numbers which fall into one or the other of two types. The types are TYPE I: pairs of consecutive powerful numbers one of which is a perfect square, and TYPE II: pairs of consecutive powerful numbers neither of which is a perfect square. He showed an infinity of cases of TYPE I by applying theory of the Pell equation.

The first purpose of this paper is to elaborate on Golomb's findings for TYPE I, through theory of the Pell equation, to give all cases of that type. Then, built on that theory, the second purpose is to formulate all pairs of consecutive powerful numbers of TYPE II, through certain solutions of another Diophantine equation.

2. CONSECUTIVE POWERFUL NUMBER PAIRS OF TYPE I

Consecutive powerful number pairs of TYPE I correspond to certain numbers satisfying the Pell equation

$$(1) \qquad X^2 - DY^2 = \pm 1 ,$$

where D is a given positive integer not a perfect square.

It is convenient to make the following definitions

Definition. The number $x + y\sqrt{D}$ is a solution of (1) if $x = X$ and $y = Y$ are integers satisfying (1).

Definition. A positive solution of (1) is a solution $x + y\sqrt{D}$ of (1) in which both integers x and y are positive.

Although at times we will consider solutions in which x or y may be negative, our main concern is with positive solutions. At all times our "integers" are assumed to be rational integers.

Definition. The positive solution $x + y\sqrt{D}$ of (1) in which X and Y have their least values is the fundamental solution of (1).

The fact that powers of the fundamental solution of (1) generate all positive solutions is well known [2] and is given here without proof in the following

Theorem 2.1. Solutions of the Pell equation (1) may be formulated by the following cases. (1) If equation (1) with the minus sign is not solvable, let $x + y\sqrt{D}$ be the fundamental solution of (1) with the plus sign. Then all positive solutions of the latter equation are given by

$$(2) \qquad x_i + y_i\sqrt{D} = (x + y\sqrt{D})^i$$

for positive integers i, and where $x_1, y_1 = x, y$. (2) If equation (1) with the minus sign is solvable and has fundamental solution $x + y\sqrt{D}$, then all its positive solutions are given by (2) for odd positive integers i. In this case the fundamental solution of (1) with the plus sign is $(x + y\sqrt{D})^2$, and all its positive solutions are given by $x_{2i} + y_{2i}\sqrt{D} = [(x + y\sqrt{D})^2]^i$ for positive integers i.

For purposes of this paper it is convenient to write solutions in the form $x_j + y_j\sqrt{D}$, where j is the exponent of the power of the fundamental solution and not necessarily the ordinal number of the solution in sequence. Without loss of generality we may assume that D is square-free.

In equation (1), for the two consecutive integers X^2 and DY^2, our desire that the latter be a powerful number motivates the following

Definition. A solution $u + v\sqrt{D}$ of equation (1) has property Q if for p a prime, $p \mid D$ implies $p \mid v$.

Definition. The least solution with property Q is the positive solution $u + v\sqrt{D}$ with property Q of (1), in which integers u and v have their least values.

Now if equation (1) with the minus sign has a solution with property Q, then D must be odd. For if $X^2 - D^2 = -1$ is solvable with D even, then no solution has property Q since $4 \nmid X^2 + 1 = DY^2$ for any integer X.

If equation (1) has the plus sign and if D is even, let $x + y\sqrt{D}$ be any solution. Then since x is odd

$$8 \mid (x - 1)(x + 1) = Dy^2 ,$$

and $2 \mid y$ since D is square-free. So here $2 \mid D$ implies $2 \mid y$.

Now in the expansion of (2) we see that for any i

$$(3) \qquad\qquad y_i = ix^{i-1}y + \binom{i}{3} x^{i-3}y^3 D + \cdots ,$$

where $y \mid y_i$, and D is a factor of every term on the right except the first. If the fundamental solution $x + y\sqrt{D}$ of (1) has property Q, then it is clear from (3) that all positive solutions have property Q. Otherwise, if we take $i = \Pi p_j$ the product of distinct odd primes p_j such that $p_j \mid D$ but $p_j \nmid y$, then we see from (3) that the solution $x_i + y_i\sqrt{D}$ has property Q. Moreover, this is the least solution with property Q since $(x, D) = 1$ and since for any $h < i$ there is at least one of the primes p_j such that $p_j \mid D$ but $p_j \nmid y_h$ in the solution $x_h + y_h\sqrt{D}$ of (1) as given by (2). We have proved the following

Theorem 2.2. For either choice of sign, let equation (1) have fundamental solution $x + y\sqrt{D}$. Then

(1). If (1) has the minus sign, and if D is even, no solutions have property Q. In all other cases (1) has a solution with property Q.

(2). If the fundamental solution has property Q, then all positive solutions have property Q.

(3). If the fundamental solution does not have property Q, then the least solution with property Q, when it exists, is the number $(x + y\sqrt{D})^i$, where the integer i is the product of the distinct odd primes dividing D but not dividing the integer y.

Before considering all solutions of (1) with property Q, we need two lemmas.

Lemma 2.3. Let $\alpha = x + y\sqrt{D}$ and $\beta = u + v\sqrt{D}$ be any solutions with property Q of equation (1) with either choice of sign for each solution. Then

(1). The product $\gamma = \alpha\beta = (xu + yvD) + (xv + yu)\sqrt{D}$ is a solution of (1) with the plus sign if α and β are solutions of (1) with the same sign, or γ is a solution of (1) with the minus sign for α and β solutions of (1) with opposite sign.

(2). The product solution γ has property Q.

Proof. The first conclusion follows from

$$(xu + yvD)^2 - D(xv + yu)^2 = (x^2 - Dy^2)(u^2 - Dv^2) = (\pm 1)(\pm 1) = \pm 1 .$$

Now by property Q for α and β if a prime $p \mid D$, then $p \mid y$ and $p \mid v$, so $p \mid (xv + yu)$ and γ has property Q.

Lemma 2.4. Let (1) with the minus sign be solvable with fundamental solution $x + y\sqrt{D}$ and have $x_i + y_i\sqrt{D}$ the least solution with property Q. Let $x_L + y_L\sqrt{D}$ be the least solution with property Q of (1) with the plus sign. Then $x_L + y_L\sqrt{D} = (x_i + y_i\sqrt{D})^2$.

Proof. By Theorem 2.1 the fundamental solution of (1) with the plus sign is

$$(x + y\sqrt{D})^2 = (x^2 + Dy^2) + 2xy\sqrt{D} .$$

By Theorem 2.2, D is odd, so $(2x, D) = 1$. Then the prime divisors of D not dividing $2xy$ are the same as those dividing D and not y for the least solution of (1) with the minus sign which has property Q. So that by Theorem 2.1 and by conclusion 3 of Theorem 2.2, we have

$$x_L + y_L \sqrt{D} = [(x + y\sqrt{D})^2]^i = [(x + y\sqrt{D})^i]^2 = (x_i + y_i\sqrt{D})^2 .$$

Theorem 2.5. Let $x_i + y_i \sqrt{D}$ be the least solution with property Q, when it exists, of equation (1). Then the formula

(4) $$x_{ih} + y_{ih} \sqrt{D} = (x_i + y_i \sqrt{D})^h$$

gives all positive solutions with property Q of equation (1) with the plus sign for positive integers h, and gives all such solutions of (1) with the minus sign for odd positive integers h.

Proof.   For either choice of sign in equation (1), since integers x_i and y_i are positive, by repeated applications of Lemma 2.3, we see that (4) always gives solutions as described in the statement of this theorem.

We now show that (4) gives all of the positive solutions with property Q of (1) for each choice of sign. Suppose that (1), for some choice of sign, has a positive solution $u + v\sqrt{D}$, which has property Q and which is not given by formula (4). Then for some positive integer h, we have

(5) $$(x_i + y_i\sqrt{D})^h < u + v\sqrt{D} < (x_i + y_i\sqrt{D})^{h+\epsilon} .$$

If equation (1) has the plus sign, then $\epsilon = 1$, or if equation (1) has the minus sign, then $\epsilon = 2$ with h odd; and respectively the number

$$x_i - y_i\sqrt{D} = \pm 1/(x_i + y_i\sqrt{D})$$

is positive or negative. For either case, multiplying inequalities (5) by $\pm(x_i - y_i\sqrt{D})^h$, whichever is positive, gives

$$1 < \alpha = (\pm u \pm v\sqrt{D})(x_i - y_i\sqrt{D})^h < \beta = (x_i + y_i\sqrt{D})^\epsilon .$$

For both of these cases, by Lemma 2.3, the number $\alpha = w + z\sqrt{D}$ is a solution with property Q of (1) with the plus sign, for some integer pair w, z. Substituting for α in the last inequalities, we get

(6) $$1 < w + z\sqrt{D} < \beta = (x_i + y_i \sqrt{D})^\epsilon ,$$

and the inequalities

(7) $$0 < w - z\sqrt{D} = 1/(w + z\sqrt{D}) < 1$$

imply that both integers w and z are positive. So $w + z\sqrt{D}$ is a positive solution of (1) with the plus sign. But in (6) we have

$$w + z\sqrt{D} < \beta = (x_i + y_i \sqrt{D})^\epsilon ,$$

where for $\epsilon = 1$ or for $\epsilon = 2$, by Lemma 2.4, the number β is the least solution with property Q of (1) with the plus sign, a contradiction. This completes the proof that (4) gives all positive solutions with property Q of equation (1).

Since equation (1) solutions $(x_i + y_i \sqrt{D})^h$ with property Q correspond to consecutive powerful numbers x_{ih}^2, Dy_{ih}^2 we have thus accounted for all consecutive powerful number pairs of TYPE I.

EXAMPLES. (1) The fundamental solution $24335 + 3588\sqrt{46}$ of $X^2 - 46Y^2 = 1$ has property Q, and hence all positive solutions have property Q. The corresponding powerful numbers are 592, 192, 225 and 592, 192, 224 or $(24335)^2$ and $(2^5)(3^2)(13^2)(23^3)$. (2) The fundamental solution of $X^2 - 6Y^2 = 1$ is $5 + 2\sqrt{6}$. The solution

$$(5 + 2\sqrt{6})^3 = 485 + 198\sqrt{6}$$

and all its powers have property Q. This solution leads to powerful numbers 235, 225 and 235, 224 or $(485)^2$ and $(2^3)(3^5)(11^2)$. (3) The fundamental solution of $X^2 - 5Y^2 = -1$ is $2 + \sqrt{5}$. The solution

$$(2 + \sqrt{5})^5 = 682 + 305\sqrt{5}$$

and its odd powers have property Q. This solution leads to the powerful numbers 465,124 and 465,125 or $(682)^2$ and $(5^3)(61^2)$.

3. CONSECUTIVE POWERFUL NUMBER PAIRS OF TYPE II

Consecutive powerful number pairs of this type correspond to certain numbers satisfying the Diophantine equation

(8) $$mX^2 - nY^2 = \pm 1,$$

where m and n are given positive integers, and neither is a perfect square.

Our development of the theory of this equation proceeds along lines similar to that of the Pell equation (1). We begin by making the following definitions

Definition. The number $x\sqrt{m} + x\sqrt{n}$ is a solution of (8) if $x = X$ and $y = Y$ are integers satisfying (8).

Definition. A positive solution of (8) is a solution $x\sqrt{m} + y\sqrt{n}$ of (8) in which both integers x and y are positive.

As before, although some solutions under consideration may have negative x or y, our main concern is with positive solutions.

REMARK. If $x\sqrt{m} + y\sqrt{n}$ and $x'\sqrt{m} + y'\sqrt{n}$ are positive solutions of (8), then it is easily seen that the inequalities

$$x < x', \qquad y < y' \quad \text{and} \quad x\sqrt{m} + y\sqrt{n} < x'\sqrt{m} + y'\sqrt{n}$$

are equivalent. So among all the positive solutions, there is one solution in which both x and y have their least values.

Definition. The smallest solution of (8) is the positive solution $x\sqrt{m} + y\sqrt{n}$ in which both integers X and Y have their least values.

Analogous to Theorem 2.1 for solutions of the Pell equation is the theorem we now state without proof [3, Theorem 9] for solutions of equation (8)

Theorem 3.1. If equation (8) has smallest solution $x\sqrt{m} + y\sqrt{n}$, then all positive solutions of (8) are given by

$$(9) \qquad x_i\sqrt{m} + y_i\sqrt{n} = (x\sqrt{m} + y\sqrt{n})^{2i+1}$$

for non-negative integers i, and where $x_0, y_0 = x, y$.

Without loss of generality we may assume that integers m and n are square-free. Moreover, our desire that consecutive integers mX^2 and nY^2 be a powerful number pair of TYPE II motivates the following

Definition. A solution $u\sqrt{m} + v\sqrt{n}$ of (8) has property Q if for p a prime, $p \mid mn$ implies $p \mid uv$.

Note that since $(mu, nv) = 1$, this definition is equivalent to saying that the prime divisors of m divide u, and those dividing n divide v.

Definition. The least solution with property Q of equation (8) is the positive solution $u\sqrt{m} + v\sqrt{n}$ with property Q of (8), in which integers u and v have their least values.

Now from (9) we get x_i and y_i in the following expressions

$$(10) \qquad x_i = m^i x^{2i+1} + \cdots + \binom{2i+1}{3} mn^{i-1} x^3 y^{2i-2} + (2i+1)n^i xy^{2i}, \text{ and}$$

$$y_i = (2i+1)m^i x^{2i} y + \binom{2i+1}{3} m^{i-1} nx^{2i-2} y^3 + \cdots + n^i y^{2i+1}.$$

Note that x_i and y_i have m and n respectively as a factor of every term except one, the term in each case having the odd positive integer $2i+1$ as a factor. Note also that $(mx, ny) = 1$, $x \mid x_i$, and $y \mid y_i$.

If one of m or n, say m, is even, we see in (10) that x_i is even for all i, if and only if x is even. Similarly, when n is even, y_i is even if and only if y is even. So for the possible prime divisor 2 of m or n, solutions with property Q of (8) depend solely on the parity of integers x or y respectively of the smallest solution $x\sqrt{m} + y\sqrt{n}$.

Now from (10) if the smallest solution $x\sqrt{m} + y\sqrt{n}$ of (8) has property Q, then all positive solutions have property Q since for p a prime, $p \mid mn$ implies $p \mid xy \mid x_i y_i$.

If the smallest solution $x\sqrt{m} + y\sqrt{n}$ of (8) does not have property Q, then for the odd integer $2i+1$ in (9) and (10), take

$$(11) \qquad 2i+1 = \Pi p_j$$

the product of distinct odd primes p_j such that $p_j \mid mn$ but $p_j \nmid xy$, and the solution $x_i\sqrt{m} + y_i\sqrt{n}$ as given in (9) and (10) has property Q. Moreover, this solution obtained in (11) is the least solution with property Q of (8). This is due to the fact that a positive solution $x_h\sqrt{m} + y_h\sqrt{n}$, with $h < i$, corresponds in (9) and (10) to an exponent $2h+1$ such that $2h+1$ and hence at least one of x_h or y_h are not divisible by some prime divisor p_j in (11) of mn.

We have proved the following

Theorem 3.2. Let equation (8) have smallest solution $x\sqrt{m} + y\sqrt{n}$. Then (1) If m (or n) is even and if x (or y) is odd, then no solutions have property Q. In all other cases (8) has a solution with property Q. (2) If the smallest solution has property Q, then all positive solutions have property Q. (3) If the smallest solution does not have property Q, then the least solution with property Q, when it exists, is the solution $x_i\sqrt{m} + y_i\sqrt{n}$ given by (9) for which the exponent $2i + 1$ is the product of the distinct odd primes dividing mn but not dividing xy.

Throughout the remainder of this discussion we will frequently be concerned with the Pell equation which we write in the form

(12) $$R^2 - mnS^2 = 1,$$

and to which the same theory and definitions apply as to the Pell equation (1), since the product mn is square-free.

Before discussing all solutions with property Q of (8), we need the following two lemmas

Lemma 3.3. If $a = u\sqrt{m} + v\sqrt{n}$ is a solution with property Q of (8), and $\beta = r + s\sqrt{(mn)}$ is a solution of (12) with property Q (as defined for that equation), then the product

$$\gamma = a\beta = (ur + nvs)\sqrt{m} + (vr + mus)\sqrt{n}$$

is a solution with property Q of (8).

Proof. $(mu^2 - nv^2)(r^2 - mns^2) = (\pm 1)(1) = m(ur + nvs)^2 - n(vr + mus)^2 = \pm 1.$

So γ is a solution of (8). Now by the properties Q for a and β respectively, if p is a prime and $p \mid m$, then $p \mid (ur + nvs)$. Similarly, if a prime $q \mid n$, then $q \mid (vr + mus)$. So γ has property Q for equation (8).

Lemma 3.4. If

$$a = u\sqrt{m} + v\sqrt{n} \qquad \text{and} \qquad \beta = u'\sqrt{m} + v'\sqrt{n}$$

are solutions with property Q of equation (8), then the product

$$\gamma = a\beta = (muu' + nvv') + (uv' + u'v)\sqrt{(mn)}$$

is a solution with property Q of (12).

Proof. $(mu^2 - nv^2)(mu'^2 - nv'^2) = (\pm 1)(\pm 1) = (muu' + nvv')^2 - mn(uv' + u'v)^2 = 1,$

and γ is a solution of (12). Now by property Q for a and β solutions of (8), the prime divisors of m divide both u and u', and prime divisors of n divide both v and v'. So if a prime $p \mid mn$, then $p \mid (uv' + u'v)$, and γ has property Q for (12).

Theorem 3.5. Let $x_i\sqrt{m} + y_i\sqrt{n}$ be the least solution with property Q of (8), when it exists. Then all positive solutions of (8) with property Q are given by the formula

(13) $$x_H\sqrt{m} + y_H\sqrt{n} = (x_i\sqrt{m} + y_i\sqrt{n})^{2h+1}$$

for non-negative integers h.

Proof.   The subscript $H = 2ih + i + h$, by Theorem 3.1. Since $x_i\sqrt{m} + y_i\sqrt{n}$ is the least solution with property Q of (8), then by Lemmas 3.3 and 3.4 formula (13) gives a positive solution with property Q of (8) for every non-negative integer h.

Now suppose equation (8) has a positive solution $w\sqrt{m} + z\sqrt{n}$, with property Q, which is not given by (13). Then for some non-negative integer h, we have

(14) $$(x_i\sqrt{m} + y_i\sqrt{n})^{2h+1} < w\sqrt{m} + z\sqrt{n} < (x_i\sqrt{m} + y_i\sqrt{n})^{2h+3}.$$

The number

$$x_i\sqrt{m} - y_i\sqrt{n} = \pm 1/(x_i\sqrt{m} + y_i\sqrt{n})$$

is positive or negative, respectively, according as equation (8) has the plus or minus sign. For either case, multiplying inequalities (14) by $\pm(x_i\sqrt{m} - y_i\sqrt{n})^{2h+1}$ whichever is positive, we get

(15) $$1 < a = (\pm w\sqrt{m} \pm z\sqrt{n})(x_i\sqrt{m} - y_i\sqrt{n})^{2h+1} < \beta = (x_i\sqrt{m} + y_i\sqrt{n})^2$$

By Lemma 3.4, and since integers x_i and y_i are positive, the number $\beta = (x_i\sqrt{m} + y_i\sqrt{n})^2$ is a positive solution with property Q of (12). We will now show that β is the least solution with property Q of (12). Suppose on the contrary

that the number $r_j + s_j \sqrt{(mn)}$ is the least solution with property Q of (12). By Theorem 2.1,

$$r_j + s_j \sqrt{(mn)} = [r + s\sqrt{(mn)}]^j,$$

where $r + s\sqrt{(mn)}$ is the fundamental solution of (12), for some positive integer j. By Theorem 2.2., the integer j is odd, and by [3, Theorem 6] the number

$$r + s\sqrt{(mn)} = (x\sqrt{m} + y\sqrt{n})^2,$$

where $x\sqrt{m} + y\sqrt{n}$ is the smallest solution of (8). A substitution gives

$$r_j + s_j \sqrt{(mn)} = [(x\sqrt{m} + y\sqrt{n})^2]^j = [(x\sqrt{m} + y\sqrt{n})^j]^2.$$

By Theorem 3.1, and since $j = 2k + 1$ is odd,

$$(x\sqrt{m} + y\sqrt{n})^j = (x\sqrt{m} + y\sqrt{n})^{2k+1} = x_k \sqrt{m} + y_k \sqrt{n},$$

the kth positive solution of (8). Then

$$r_j + s_j \sqrt{(mn)} = (x_k \sqrt{m} + y_k \sqrt{n})^2 = mx_k^2 + ny_k^2 + 2x_k y_k \sqrt{(mn)}.$$

Now since $r_j + s_j \sqrt{(mn)}$ has property Q for (12) and since $s_j = 2x_k y_k$ it follows that if a prime $p \mid mn$, then $p \mid 2x_k y_k$. In fact if a prime $p \mid mn$, then $p \mid x_k y_k$. This is obvious if p is an odd prime. Since equation (8) is assumed to have a solution with property Q, then by Theorem 3.2 if $2 \mid mn$, then $2 \mid xy \mid x_k y_k$ for all k, and where integers x, y are those of the smallest solution $x\sqrt{m} + y\sqrt{n}$ of (8). So the positive solution $x_k \sqrt{m} + y_k \sqrt{n}$ of (8) has property Q.

Then if

$$r_j + s_j \sqrt{(mn)} < \beta = (x_i \sqrt{m} + y_i \sqrt{n})^2, \quad \text{or} \quad (x_k \sqrt{m} + y_k \sqrt{n})^2 < (x_i \sqrt{m} + y_i \sqrt{n})^2,$$

it follows that

$$x_k \sqrt{m} + y_k \sqrt{n} < x_i \sqrt{m} + y_i \sqrt{n}$$

the least solution with property Q of (8), a contradiction. Thus we have shown the number $\beta = (x_i \sqrt{m} + y_i \sqrt{n})^2$ of inequalities (15) to be the least solution with property Q of (12).

Now consider the number α of inequalities (15). By Lemmas 3.3 and 3.4 the number $\alpha = u + v\sqrt{(mn)}$ is a solution with property Q of (12), for some pair of integers u and v. So that inequalities (15) become

(16) $$1 < u + v\sqrt{(mn)} < (x_i \sqrt{m} + y_i \sqrt{n})^2.$$

Then since $u + v\sqrt{(mn)}$ is a solution of (12), the inequalities

$$0 < u - v\sqrt{(mn)} = 1/[u + v\sqrt{(mn)}] < 1$$

imply that integers u and v are both positive, and $u + v\sqrt{(mn)}$ is a positive solution of (12).

We have shown that the existence of a positive solution $w\sqrt{m} + z\sqrt{n}$ with property Q of equation (8), which is not given by formula (13), implies the existence of a positive solution $u + v\sqrt{(mn)}$, with property Q, of equation (12), which by (16) is less than $(x_i \sqrt{m} + y_i \sqrt{n})^2$ the least solution with property Q of (12), a contradiction. This completes the proof that formula (13) gives all positive solutions with property Q of equation (8).

Since equation (8) solutions $(x_i \sqrt{m} + y_i \sqrt{n})^{2h+1}$ with property Q correspond to consecutive powerful numbers $mx_H^2 \; ny_H^2$ we have thus accounted for all consecutive powerful number pairs of TYPE II and hence, with Section 2 of this paper, for all pairs of consecutive powerful numbers.

EXAMPLE. The equation $7X^2 - 3Y^2 = 1$ has smallest solution $2\sqrt{7} + 3\sqrt{3}$. The solution

$$(2\sqrt{7} + 3\sqrt{3})^7 = 2,637,362\sqrt{7} + 4,028,637\sqrt{3}$$

and all its odd powers have property Q. This solution corresponds to the following consecutive powerful number pair of TYPE II,

$$48,689,748,233,308 = 7(2,637,362)^2 = (2^2)(7^3)(13^2)(43^2)(337^2)$$

and

$$48,689,748,233,307 = 3(4,028,637)^2 = (3^3)(139^2)(9661^2).$$

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