ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, 709 Solano Dr., S. E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-334 Proposed by Phil Mana, Albuquerque, New Mexico.

Are all the terms prime in the sequence 11, 17, 29, 53, ⋯ defined by $u_0 = 11$ and $u_{n+1} = 2u_n - 5$ for $n \geqslant 0$?

B-335 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain a closed form for

$$\sum_{i=0}^{n-k} (F_{i+k}L_i + F_iL_{i+k}).$$

B-336 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let $Q_0 = 1 = Q_1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$. Show that $2(Q_{2n}^2 - 1)$ is a perfect square for n = 1, 2, 3, ⋯.

B-337 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Show that there are infinitely many points with both x and y rational on the ellipse $25x^2 + 16y^2 = 82$.

B-338 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Let k and n be positive integers. Let $p = 4k + 1$ and let h be the largest integer with $2h + 1 \leqslant n$. Show that

$$\sum_{j=0}^{h} p^j \binom{n}{2j+1}$$

is an integral multiple of 2^{n-1}.

B-339 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Establish the validity of E. Cesàro's symbolic Fibonacci-Lucas identity $(2u+1)^n = u^{3n}$; after the binomial expansion has been performed, the powers of u are used as either Fibonacci or Lucas subscripts. (For example, when $n = 2$ one has both

$$4F_2 + 4F_1 + F_0 = F_6 \qquad \text{and} \qquad 4L_2 + 4L_1 + L_0 = L_6\,.)$$

SOLUTIONS

SPECIAL BINOMIAL COEFFICIENTS

B-310 Proposed by Daniel Finkel, Brooklyn, New York.

Find some positive integers n and r such that the binomial coefficient $\binom{n}{r}$ is divisible by $n + 1$.

Solution by David Singmaster, Polytechnic of the South Bank, London, England.

For $n \leq 100$, I find the following solutions for $(n+1) \mid \binom{n}{r}$, with $2r \leq n$;

$$n = 29, \quad r = 6, 7, 14;$$
$$n = 59, \quad r = 12, 13, 14, 15;$$
$$n = 69, \quad r = 21, 22, 23, 24;$$
$$n = 83, \quad r = 36, 37, 38, 39, 40, 41;$$
$$n = 89, \quad r = 15, 18, 19, 20, 21, 22, 23, 40, 41, 42, 44.$$

One can show that $n + 1$ must have at least three prime factors.

Also solved by the Proposer.

A NONHOMOGENEOUS RECURRENCE

B-311 Proposed by Jeffrey Shallit, Wynnewood, Pennsylvania.

Let k be a constant and let $\{a_n\}$ be defined by

$$a_n = a_{n-1} + a_{n-2} + k, \qquad a_0 = 0, \qquad a_1 = 1.$$

Find

$$\lim_{n \to \infty} (a_n / F_n).$$

Solution by Graham Lord, Université Laval, Québec.

With $a_0 = 0$ and $a_1 = 1$ then $a_n = F_n + (F_{n+1} - 1)k$ (use induction) and so the limit is $1 + \alpha k$.

Also solved by George Berzsenyi, Paul S. Bruckman, Charles Chouteau, Herta T. Freitag, Ralph Garfield, Frank Higgins, Harvey J. Hindin, Mike Hoffman, John W. Milsom, C.B.A. Peck, A. G. Shannon, Martin C. Weiss, Gregory Wulczyn, David Zeitlin, Larry Zimmerman, and the Proposer.

DOUBLY-TRUE FIBONACCI ALPHAMETIC

B-312 Proposed by J. A. H. Hunter, Fun with Figures, Toronto, Ontario, Canada.

Solve the doubly-true alphametic

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    O N E
    O N E
    O N E
    T W O
T H R E E
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E I G H T
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Unity is not normally considered so, but here our ONE is prime!

Solution by Charles W. Trigg, San Diego, California.

$O \neq$ zero, $T + 1 = E$, and ONE is prime, so $E = 3$, 7, or 9. Then $4E + O = T + 10k$.

If $E = 3$, then O = zero, which is not acceptable.

If $E = 9$, then $T = 8$, $O = 2$, $H = 7$, and the sum of the digits in the hundreds' column is <30. Hence, $E \neq 9$.

If $E = 7$, then $T = 6$, and $O = 8$, whereupon $N = 2$ or 5, since ONE is prime. But if $N = 2$, then $I = 2$. Consequently, $N = 5$, $H = 9$, $I = 2$, $W = 4$, $R = 1$, and $G = 3$.

The unique reconstruction of the addition is:

$$857 + 857 + 857 + 648 + 69177 = 72396.$$

Also solved by Hai Vo Ba, Richard Blazej, Paul S. Bruckman, Madeleine Hatzenbuehler and George Berzsenyi (jointly), John W. Milsom, C.B.A. Peck, A. G. Shannon, Martin C. Weiss, and the Proposer.

EXPONENTIATING LUCAS INTO FIBONACCI

B-313 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Let

$$M(x) = L_1x + (L_2/2)x^2 + (L_3/3)x^3 + \cdots.$$

Show that the Maclaurin series expansion for $e^{M(x)}$ is

$$F_1 + F_2x + F_3x^2 + \cdots.$$

1. Solution by Graham Lord, Universite Laval, Québec.

If L_n is replaced by $\alpha^n + \beta^n$ then $M(x)$ becomes

$$-\ln(1-\alpha x)(1-\beta x)$$

which is the same as

$$-\ln(1-x-x^2).$$

Hence $e^{M(x)}$ is $1/(1-x-x^2)$, that is

$$F_1 + F_2x + F_3x^2 + \cdots.$$

2. Solution by Martin C. Weiss, San Jose, California.

$$M'(x) = L_1 + L_2x + L_3x^2 + \cdots = (1+2x)/(1-x-x^2).$$

Integrating, $M(x) = -\ln(1-x-x^2)$. Hence,

$$e^{M(x)} = 1/(1-x-x^2) = F_1 + F_2x + F_3x^2 + \cdots.$$

Also solved by Paul S. Bruckman, Charles Chouteau, Herta T. Freitag, Ralph Garfield, Harvey J. Hindin, Mike Hoffman, A. G. Shannon, Sahib Singh, David Zeitlin, and the Proposer.

LUCAS NUMBERS ENDING IN THREE

B-314 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $L_{2p^k} \equiv 3 \pmod{10}$ for all primes $p \geqslant 5$.

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle, Illinois.

For all primes $p \geqslant 5$, $p \equiv \pm 1 \pmod 6$. Hence, for all natural k, $p^k \equiv \pm 1 \pmod 6$, which implies $2p^k \equiv \pm 2 \pmod{12}$.

If we now write down the Lucas sequence (mod 10), we readily find that the cycle has length 12 (i.e. $L_{m+12} \equiv L_m \pmod{10}$, $\forall\, m$); it is also easy to observe that

$$L_n \equiv 3 \pmod{10} \quad \text{iff} \quad m \equiv \pm 2 \pmod{12}.$$

Combining this with the first result above, it follows that $L_{2p^k} \equiv 3 \pmod{10}$, for all prime $p \geqslant 5$.

Also solved by Frank Higgins, Graham Lord, A. G. Shannon, Martin C. Weiss, Gregory Wulczyn, David Zeitlin, and the Proposer.

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