ON A GENERALIZATION OF THE FIBONACCI NUMBERS USEFUL IN MEMORY ALLOCATION SCHEMA; OR ALL ABOUT THE ZEROES OF $Z^k - Z^{k-1} - 1,\ k > 0$

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ABSTRACT

A generalization of the Fibonacci numbers arises in the theory of dynamic storage allocation schema. The associated linear recurrence relation involves the polynomial $Z^k - Z^{k-1} - 1,\ k \geqslant 1$. A theorem is proven showing that all the zeroes of this polynomial lie in the intersection of two annuli.

Complete information about the sequence then follows, e.g., expressing the elements in terms of certain sums of binomial coefficients and sums of powers of roots, limits of quotients of terms, and limits of roots. Tables useful for storage design are included.

A certain linear recurrence relation arises in the theory of memory allocation schema which generalizes the linear recurrence defining the Fibonacci numbers. The generalized numbers may be expressed as the coefficients of a rational generating function where the denominator of the rational function involves the trinomial $Z^k - Z^{k-1} - 1$. From this fact follows two expressions for the numbers themselves, one in terms of linear combinations of the powers of the roots of the trinomial, and another expression giving the numbers as sums of binomial coefficients which lie on a line of rational slope falling across Pascal's triangle. The former expression gives complete information on the limit of successive quotients. This latter data depends upon the location of the roots of this trinomial: all complex zeroes lie in the intersection of two annuli in the complex plane. See Table 1 and Figure 1 for explicit numbers and visulization of the following central theorems.

Theorem A. Let $k \geqslant 1$. All of the k zeroes of $Z^k - Z^{k-1} - 1$ are distinct and lie in the intersection of the two annuli

$$\lambda_0 \leqslant |Z| \leqslant \lambda_1 \quad \text{and} \quad \lambda_1 - 1 \leqslant |Z - 1| \leqslant 1 + \lambda_0,$$

where $\lambda_\epsilon = \lambda_\epsilon(k)$ is the largest (positive) real solution of

$$r^k + (-1)^\epsilon r^{k-1} - 1 = 0, \quad \epsilon = 0, 1, \quad 0 < \lambda_0 < 1 < \lambda_1 < 2.$$

Table 2 gives approximate values of these $\lambda_\epsilon = \lambda_\epsilon(k),\ k = 1, 2, \cdots, 20, 100.$

Theorem B. Let $k \geqslant 1$. Define $f_{k,n} = f_{k,n-1} + f_{k,n-k}\,;\ f_{k,j} = 0, j < k;\ f_{k,k} = 1.$ Then

$$\lim_{n\to\infty} \frac{f_{k,n+1}}{f_{k,n}} = \lambda_1(k) \quad \text{and} \quad \lim_{n\to\infty} \lambda_1(k) = 1.$$

The proofs of these theorems depend upon two sequences of lemmas, those bearing more directly upon Theorem A or B; we number the lemmas accordingly.

Lemma A1. Let $p(Z) = Z^k - Z^{k-1} - 1,\ k \geqslant 1$. None of the zeroes of $p(Z)$ are rational; all of the zeroes of $p^{(1)}(Z)$ are rational.

Proof. Since

$$p^{(1)}(Z) = kZ^{k-2}\left(Z - \frac{k-1}{k}\right)$$

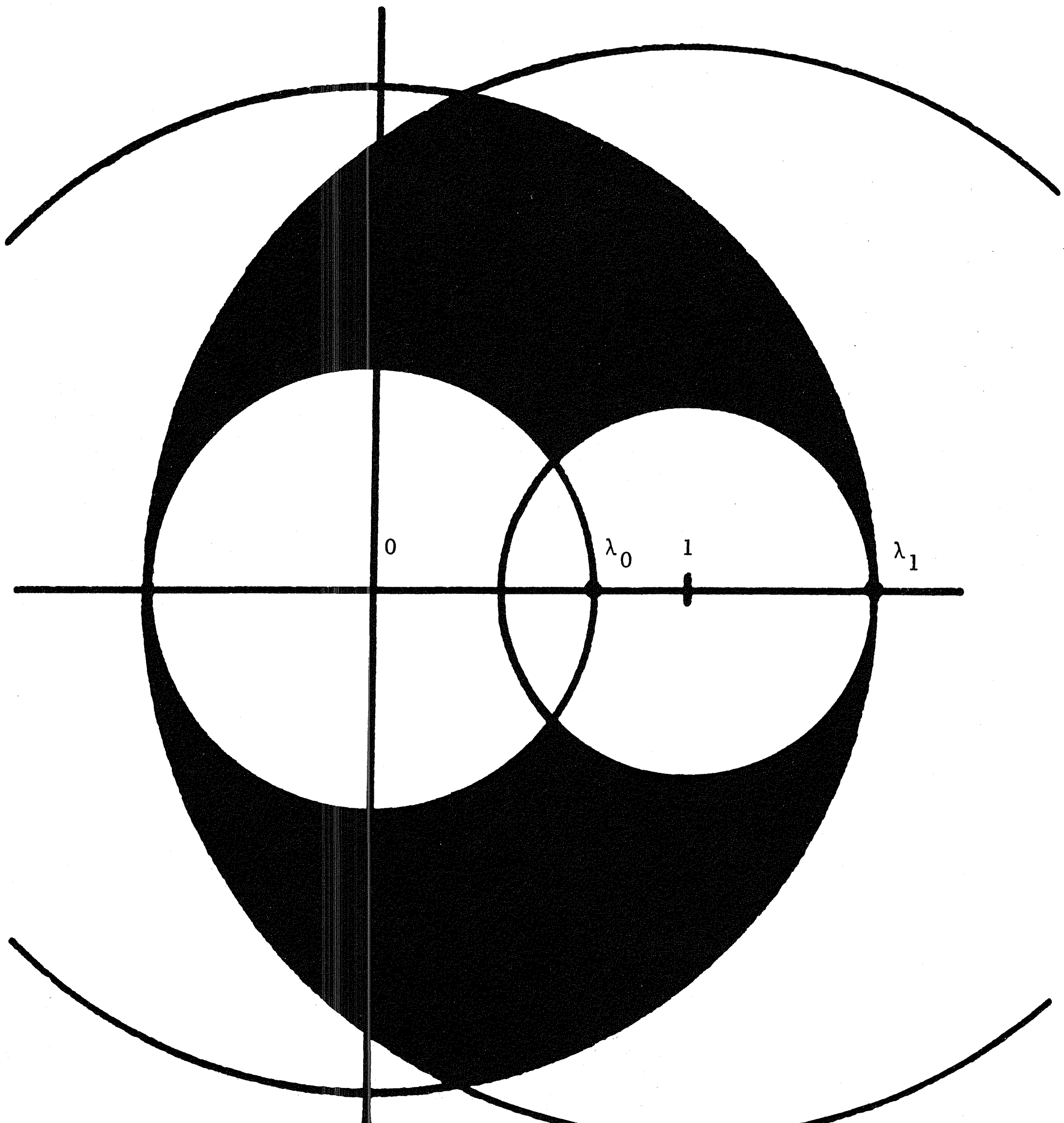

Figure 1. The Two Annuli Theorem
(The shaded region represents the region in which all of the complex zeroes of $Z^k - Z^{k-1} - 1$ must lie.)

Table 1
The Sequences $f_{k,n} = f_{k,n-1} + f_{k,n-k}$
With $f_{k,j} = 0, j < k;\ f_{k,k} = 1, k \geqslant 1$

n^k	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0	0
3	4	1	1	0	0	0	0	0	0	0	0
4	8	2	1	1	0	0	0	0	0	0	0
5	16	3	1	1	1	0	0	0	0	0	0
6	32	5	2	1	1	1	0	0	0	0	0
7	64	8	3	1	1	1	1	0	0	0	0
8	128	13	4	2	1	1	1	1	0	0	0
9	256	21	6	3	1	1	1	1	1	0	0
10	512	34	9	4	2	1	1	1	1	1	0
11	1024	55	13	5	3	1	1	1	1	1	1
12	2048	89	19	7	4	2	1	1	1	1	1
13	4096	144	28	10	5	3	1	1	1	1	1
14	8192	233	41	14	6	4	2	1	1	1	1
15	16384	377	60	19	8	5	3	1	1	1	1
16	32768	610	88	26	11	6	4	2	1	1	1
17	65536	987	129	36	15	7	5	3	1	1	1
18	131072	1597	189	50	20	9	6	4	2	1	1
19	262144	2584	277	69	26	12	7	5	3	1	1
20	524288	4181	406	95	34	16	8	6	4	2	1
21	1048576	6765	595	131	45	21	10	7	5	3	1
22	2097152	10946	872	181	60	27	13	8	6	4	2
23	4194304	17711	1278	250	80	34	17	9	7	5	3
24	8388608	28657	1873	345	106	43	22	11	8	6	4
25	16777216	46368	2745	476	140	55	28	14	9	7	5
26	33554432	75025	4023	657	185	71	35	18	10	8	6
27	67108864	121393	5896	907	245	92	43	23	12	9	7
28	134217728	196418	8641	1252	325	119	53	29	15	10	8
29	268435456	317811	12664	1728	431	153	66	36	19	11	9
30	536870912	514299	18560	2385	571	196	83	44	24	13	10
31	1073741824	832040	27201	3292	756	251	105	53	30	16	11

Table 2
$\lambda_\epsilon = \lambda_\epsilon(k)$, $\epsilon = 0, 1$ is the Largest Positive Real Root of $r^k + (-1)^\epsilon r^{k-1} - 1$.
The roots are truncated to 25 decimal places; see [3].

k	$\lambda_1(k)$	$\lambda_0(k)$
1	2.0000000000000000000000000	0.0000000000
2	1.6180339887498948482045868	0.6180339887498948482045868
3	1.4655712318767680266567312	0.7548776662466927600495088
4	1.3802775690976141156733016	0.8191725133961644396995711
5	1.3247179572447460259609088	0.8566748838545028748523248
6	1.2851990332453493679072604	0.8812714616335695944076491
7	1.2554228710768465432050014	0.8986537126286992932608757
8	1.2320546314285722959319676	0.9115923534820549186286736
9	1.2131497230596399145540815	0.9215993196339830062994303
10	1.1974914335516807746915412	0.9295701282320228642044130
11	1.1842763223508938723515139	0.9360691110777583783971914
12	1.1729507500239802071448788	0.9414696173216352043780467
13	1.1631197906692044180088153	0.9460285282856136156355381
14	1.1544935507090564328867379	0.9499283999636198830314051
15	1.1468540421995067272864110	0.9533025374016641591079826
16	1.1400339374770049101652704	0.9562505576379890668254960
17	1.1339024903348373489121350	0.9588484010075613716652026
18	1.1283559396916029856471042	0.9611549719964985735216646
19	1.1233108062463267587889592	0.9632166633389015467989664
20	1.1186991080522260494554442	0.9650705109167162350928078
100	1.034	0.9930

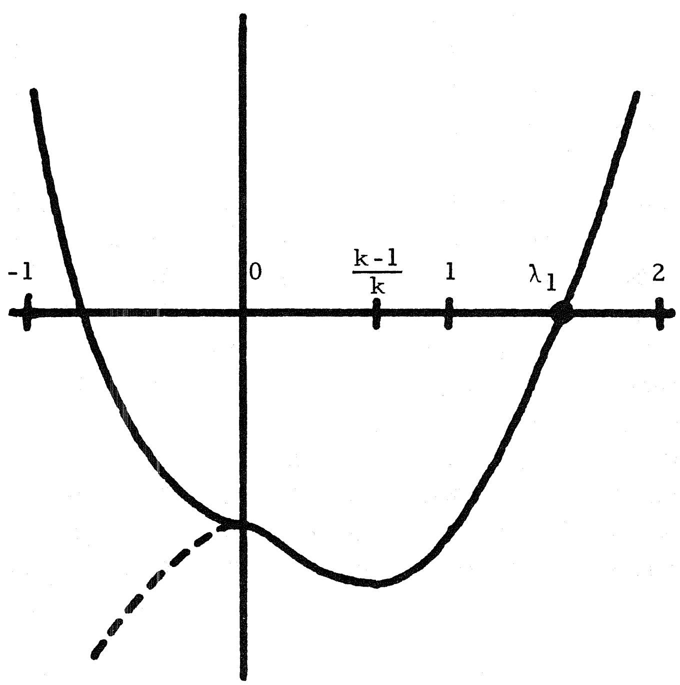

Figure 2. Combined graph of $x^k - x^{k-1} - 1 = y$ for k even and odd. There is a local minimum at $x = \frac{k-1}{k}$.

we see that the roots of $p^{(1)}(Z)$ are 0 with multiplicity $k-2$ and $(k-1)/k$ with multiplicity 1, both rational. Since $p(Z)$ is monic with integer coefficients any rational root must be a gaussian integer. From the relation $Z^{k-1}(Z-1)=1$ it is easy to infer that Z cannot be integral.

Corollary A1. Define the collection of zeroes of $p(Z)$ to be

$$Z_k = \left\{ Z \in \mathbb{C} : p(Z) = 0 \right\} = \left\{ \lambda_{k,j} ; 1 \le j \le k \right\}.$$

Then $[Z_k] = k$, i.e., the roots are distinct, and we can order them

$$|\lambda_{k,j}| \le |\lambda_{k,j+1}|, \qquad j = 1, 2, \cdots, k-1$$

with equality iff $\lambda_{k,j}$ is the complex conjugate of $\lambda_{k,j+1}$.

Proof. From Lemma A1 we have proven that $p(Z)$ and $p^{(1)}(Z)$ are relatively prime ($\mathbb{C}$ is algebraically closed) which is sufficient for the roots to be distinct. We note that in addition to nonreal complex zeroes occurring in conjugate pairs, exactly two roots are real if k is even and exactly one is real if k is odd.

Lemma A2. There exist numbers, $0 < \lambda_0 < 1 < \lambda_1 < 2$ dependent only upon k, $k > 1$, such that all of the zeroes of $p(Z) = Z^k - Z^{k-1} - 1$ lie in an annulus $\lambda_0 \le |Z| \le \lambda_1$ centered at 0 and in an annulus $\lambda_1 - 1 \le |Z-1| \le 1 + \lambda_0$ centered at 1.

Proof. Since $p(0) \ne 0$, any complex zero Z of $p(Z)$ has norm $|Z| = r > 0$ and $p(Z) = 0$ gives $|Z-1| = r^{1-k}$. Thus any zero lies on the intersection of the two circles $|Z| = r$ and $|Z-1| = r^{1-k}$ with fixed centers. There are two cases of empty intersection: one circle lying wholly inside the other. Comparing radii of these circles there will be a non-vacuous intersection if $r \le 1 + r^{1-k}$ or if $r \le \lambda_1$ where λ_1 is the largest positive root of $p(Z)$. (!). The second case of $|Z| = r$ lying inside $|Z-1| = r^{1-k}$ yields $0 \le r^k + r^{k-1} - 1$ or $r \ge \lambda_0$ where λ_0 is the largest positive root of $q(Z) = Z^k + Z^{k-1} - 1$. Locating these roots gives the inequalities above and noting that $\lambda_0^{1-k} = 1 + \lambda_0$ $\lambda_1^{1-k} = \lambda_1 - 1$ bounds the radius r^{1-k}

Corollary A2. Set $\lambda_{k,k} = \lambda_1(k) = \lambda_1$. Then $\lambda_1(k)$ is real and $|\lambda_{k,j}| < \lambda_1(k)$ for $1 \le j \le k$.

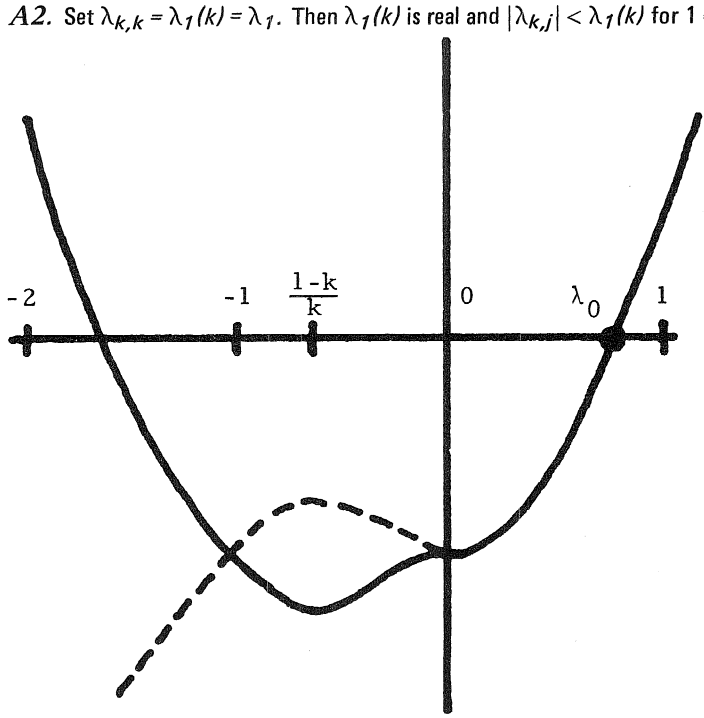

Figure 3. Combined Graph of $x^k + x^{k-1} - 1 = y$ for k Even and Odd. There is a local maximum and minimum at $x = (1-k)/k$.

Proof. λ_1 is, from the proof of Lemma A2 the largest possible real root of $p(Z)$. Note that if k is even that $-\lambda_0$ is the smallest real root of $p(Z)$.

Lemma A3. Let

$$\sum_{1 \leqslant j \leqslant k} c_j \lambda_{k,j}^n$$

be any (complex) linear combination of the n^{th} powers of the zeroes of $p(Z)$. Then, for

$$A = \sum_{1 \leqslant j \leqslant k} |C_j| \leqslant k \max_{1 \leqslant j \leqslant k} |C_j|, \qquad \left| \sum_{1 \leqslant j \leqslant k} C_j \lambda_{k,j}^n \right| \leqslant A \lambda_1^n .$$

Proof. This follows directly from Corollary A2 and the usual absolute value inequalities. This Lemma gives information on the rate of growth of the integers $f_{k,n}$.

Lemma A4. For $p_k(x) = x^k - x^{k-1} - 1$,

$$1 + \sum_{1 \leqslant j \leqslant k} p_j(x) = x^k - k.$$

Proof. The sum telescopes. The purpose of this simple Lemma is to motivate the next Lemma; the largest positive real zero of the sum is $k^{1/k}$.

Lemma A5. Let $k > 3$. Then $1 < \lambda_1(k) < k^{1/k}$.

Proof. Since $p(1) = -1$ we need only show that $p(k^{1/k}) > 0$. For $k > 3$ it is clear that

$$1 + \frac{1}{2(k-1)} < \ln k.$$

But

$$1 + \frac{1}{2(k-1)} = \frac{1}{2k} + \frac{1}{2k^2} + \frac{1}{2k^3} + \cdots > 1 + \frac{1}{2k} + \frac{1}{3k^2} + \frac{1}{4k^3} + \cdots = -k \ln\left(1 - \frac{1}{k}\right)$$

so that

$$-k \ln\left(1 - \frac{1}{k}\right) < \ln k.$$

Rewriting, we have

$$\ln\left(\frac{k-1}{k}\right) > \ln k^{-1/k},$$

exp is order preserving so that

$$1 - \frac{1}{k} > k^{-1/k}.$$

Then

$$0 < \frac{1}{k} + \frac{1}{k^{1/k}} < 1.$$

But this gives

$$0 < k - k^{1-(1/k)} - 1 = p(k^{1/k}).$$

Lemma A6. Let $k > 2$. Then $k^{-1/k} < \lambda_0(k) < 1$.

Proof. For

$$q(z) = x^k + x^{k-1} - 1, \quad q(1) = 1$$

it is sufficient to show that $q(k^{-1/k}) < 0$. It is clear that $k^{1/k} < k - 1$ for k an integer larger than two. But then $1 + k^{1/k} < k$ gives

$$0 > \frac{1}{k} + \frac{k^{1/k}}{k} - 1 = q(k^{1/k}).$$

Lemma A7. $\lim_{k \to \infty} k^{1/k} = 1.$

Proof. This follows from $\lim_{k \to \infty} (\ln k)/k = 0.$

The development concludes the proof of Theorem A and the second limit of Theorem B. We now proceed to the rationality of the generating function, the two closed form expressions for its coefficients and the limit of successive ratios.

Lemma B1. Let $f_{k,n}$ be defined as in Theorem B. For $k \geqslant 1$, the generating function for $f_{k,n}$, viz.,

(1) $$G_k(t) = \sum_{n \geqslant 0} f_{k,n} t^n$$

is a rational function of t. In fact,

(2) $$G_k(t) = \frac{t^k}{1 - t - t^k}$$

(3) $$= t^k \sum_{1 \leqslant j \leqslant k} \frac{A_{k,j}}{1 - \lambda_{k,j} t},$$

where the $\lambda_{k,j}$ are as in Corollary A1 and

(4) $$A_{k,j} = B_{k,j} \lambda_{k,j}^k$$

with

(5) $$B_{k,j} = \frac{\lambda_{k,j} - 1}{k\lambda_{k,j} - (k-1)}.$$

Proof. Given equations (2) and (3), we have

(3′) $$1 = \sum_{1 \leqslant j \leqslant k} A_{k,j} \frac{1 - t - t^k}{1 - \lambda_{k,j} t}.$$

From Lemma A1, and letting $t \to \lambda_{k,j}^{-1}$ we have

(4′) $$A_{k,j} = \frac{\lambda_{k,j}^k}{k - \lambda_{k,j}^{k-1}}$$

which, with $\lambda_{k,j}^{1-k} = \lambda_{k,j} - 1$ yields (5).

From the initial conditions, $f_{k,j} = 0,\ j < k,\ f_{k,k} = 1$ we have $f_{k,k+j} = 1,\ 0 \leqslant j < k$ by referring to the relation

(6) $$f_{k,n} = f_{k,n-1} + f_{k,n-k}.$$

Then

(7) $$G_k(t) = \sum_{k \leqslant n < 2k} f_n t^n + \sum_{n \geqslant 2k} f_n t^n$$

and

(8) $$tG_k(t) = \sum_{k < n < 2k} f_{n-1} t^n + \sum_{n \geqslant 2k} f_{n-1} t^n$$

(9) $$t^k G_k(t) = 0 + \sum_{n \geqslant 2k} f_{n-k} t^n.$$

From the relation (6) we have the equation

$$tG_k(t) - \sum_{k<n<2k} f_{n-1}t^n + t^k G_k(t) = G_k(t) - \sum_{k\leq n<2k} f_n t^n . \tag{10}$$

Isolating $G_k(t)$ and noting that

$$t^k = \sum_{k\leq n<2k} f_n t^n - \sum_{k<n<2k} f_{n-1}t^n \tag{11}$$

we have

$$G_k(t) = \frac{t^k}{1-t-t^k} \tag{12}$$

$$= \frac{t^k}{\prod_{1\leq j\leq k} (1-\lambda_{k,j}t)} , \tag{13}$$

where $\lambda_{k,j}$ are the solutions of $Z^k - Z^{k-1} = 0$. Clearly,

$$\sum_{1\leq j\leq k} \lambda_{k,j} = 1, \qquad \prod_{1\leq j\leq k} \lambda_{k,j} = (-1)^{k-1} . \tag{14}$$

Since, by Lemma A1 the $\lambda_{k,j}$ are all distinct we have the partial fractions decomposition stated in the Lemma, Eq. (3).

Lemma B2. Let $k \geq 1$.

$$f_{k,n} = \sum_{0\leq m\leq(n-k)/k} \binom{n-k-(k-1)m}{m} . \tag{15}$$

Proof. From Eq. (2) in Lemma B1 we have

$$G_k(t) = \frac{t^k}{1-(t+t^k)} \tag{16}$$

$$= t^k \sum_{s\geq 0} t^s(1+t^{k-1})^s \tag{17}$$

$$= \sum_{s\geq 0} t^{s+k} \sum_{0\leq m\leq s} \binom{s}{m} t^{(k-1)m} \tag{18}$$

$$= \sum_{s\geq 0} \sum_{0\leq m\leq s} \binom{s}{m} t^{s+k+(k-1)m} \tag{19}$$

$$= \sum_{n\geq 0} t^n \sum_{0\leq m\leq(n-k)/k} \binom{n-k-(k-1)m}{m} . \tag{20}$$

Thus (15) follows from the definition of $G_k(t)$. Note that if $k = 1$,

(21) $$f_{1,n} = \sum_{0 \le m \le n-1} \binom{n-1}{m} = 2^{n-1},$$

corresponding to summing Pascal's triangle horizontally. If $k = 2$, the case of Fibonacci numbers yields the familiar

(22) $$f_{2,n} = \sum_{0 \le m \le (n-2)/2} \binom{n-2-m}{m},$$

corresponding to summing the binomial coefficients lying upon lines of slope 1 through Pascal's triangle. In general one sums along lines of slope $k-1$. See Figure 4.

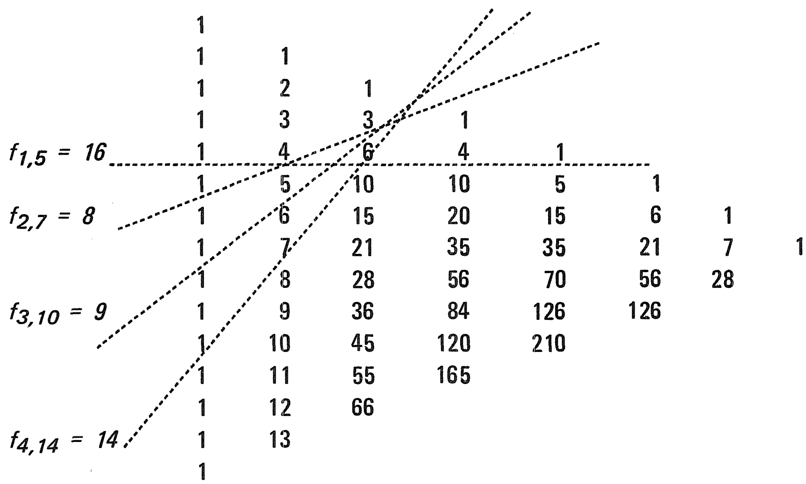

Figure 4. The Numbers $f_{k,n}$ as Sums of Binomial Coefficients Lying Upon Lines of Slope $k-1$ through Pascal's Triangle. (See Lemma B2.)

Lemma B3. Let $k \ge 1$. Then

(23) $$f_{k,n} = \sum_{1 \le j \le k} \frac{(\lambda_{k,j} - 1)}{k\lambda_{k,j} - (k-1)} \lambda_{k,j}^n,$$

where the $\lambda_{k,j}$ are the zeroes of

$$z^k - z^{k-1} - 1.$$

Proof. From Eq. (3),

(24) $$G_k(t) = t^k \sum_{1 \le j \le k} A_{k,j} \sum_{n \ge 0} \lambda_{k,j}^n t^n,$$

(25) $$= \sum_{n \ge 0} t^{n+k} \sum_{1 \le j \le k} A_{k,j} \lambda_{k,j}^n,$$

(26) $$= \sum_{n \ge k} t^n \sum_{1 \le j \le k} B_{k,j} \lambda_{k,j}^n.$$

Table 3

Real and Complex Zeroes Rounded to Five Places, $\lambda_{k,j}, j = 1, 2, \cdots, k$, of the Polynomial $Z^k - Z^{k-1} - 1$ for $k = 1, 2, \cdots, 10$ (The zeroes are listed in decreasing order of modulus. A more complete table of these roots, $k = 1, 2, \cdots, 20$ to 28 significant digits is available upon request.)

k	$\lambda_{k,k}$					
1	2.00000					
2	1.61803	−0.61803				
3	1.46557	−0.23279 ± i0.79255				
4	1.38028	0.21945 ± i0.91447	−0.81917			
5	1.32472	0.50000 ± i0.86603	−0.66236 ± i0.56228			
6	1.28520	0.67137 ± i0.78485	−0.37333 ± i0.82964	−0.88127		
7	1.25542	0.78019 ± i0.70533	−0.10935 ± i0.93358	−0.79855 ± i0.42110		
8	1.23205	0.85224 ± i0.63526	0.10331 ± i0.95648	−0.61578 ± i0.68720	−0.91159	
9	1.21315	0.90173 ± i0.57531	0.26935 ± i0.94058	−0.41683 ± i0.84192	−0.86082 ± i0.33435	
10	1.19749	0.93677 ± i0.52431	0.39863 ± i0.90691	−0.23216 ± i0.92442	−0.73720 ± i0.57522	−0.92957

Lemma B4. Fix $k \geqslant 1$. Then

$$\lim_{n \to \infty} \frac{f_{k,n+1}}{f_{k,n}} = \lambda_{k,\max}, \tag{27}$$

where $\lambda_{k,\max}$ is the largest positive real root of $Z^k - Z^{k-1} - 1$. In fact, $\lambda_{k,\max} = \lambda_{k,k}$.

Proof. From Lemma B3,

$$\frac{f_{k,n+1}}{f_{k,n}} = \frac{\sum_{1 \leqslant j \leqslant k} B_{k,j}\lambda_{k,j}^{n+1}}{\sum_{1 \leqslant j \leqslant k} B_{k,j}\lambda_{k,j}^{n}}. \tag{28}$$

Define $\lambda_{k,\max}$ to be the zero of $Z^k - Z^{k-1} - 1$ with largest absolute value. Then

$$\frac{f_{k,n+1}}{f_{k,n}} = \lambda_{k,\max} \frac{\sum_{1 \leqslant j \leqslant k} B_{k,j}\left(\frac{\lambda_{k,j}}{\lambda_{k,\max}}\right)^{n+1}}{\sum_{1 \leqslant j \leqslant k} B_{k,j}\left(\frac{\lambda_{k,j}}{\lambda_{k,\max}}\right)^{n}}. \tag{29}$$

Letting $n \to \infty$, each sum in the quotient has one or two terms depending upon whether $\lambda_{k,\max}$ is real or complex and in the latter case the limit need not exist. But from the proof of Lemma A2, $\lambda_{k,\max}$ is real and is equal to $\lambda_{k,k}$. (Each nonreal complex root has absolute value r such that $1 + r^{1-k} > r$ or $r < \lambda_{k,k} = \lambda_1(k)$.) Since

$$\lim_{n \to \infty} (\lambda_{k,j}/\lambda_{k,k})^n = \delta_k^j,$$

the Lemma follows.

Lemma A8. Let $k > 1$. Then

$$\lambda_\epsilon(k) = \lim_{n \to \infty} \mu_{\epsilon,n} \tag{30}$$

where

$$\mu_{0,n+1} = (1 + \mu_{0,n})^{1/(1-k)}, \quad \mu_{0,0} = 1 \quad \text{and} \quad \mu_{1,n+1} = 1 + \mu_{1,n}^{1-k}, \quad \mu_{1,0} = 1.$$

Proof. Clear

Lemma A9. For $k \geq 0$

(31) $$\lambda_1(k) > \lambda_1(k+1).$$

In other words $\lambda_1(k)$ converges monotonically to 1 as $k \to \infty$.

Proof. [2]. Let $r = \lambda_1(k)$, $s = \lambda_1(k+1)$. Then $r > 1$, $s > 1$, $r \neq s$, and

$$r(r^k - r^{k-1} - 1) = 0, \qquad s^{k+1} - s^k - 1 = 0.$$

Subtracting the second equation from the first and dividing through by $r - s$ we have

(32) $$\frac{(r^{k+1} - s^{k+1})}{r-s} - \frac{(r^k - s^k)}{r-s} = \frac{r-1}{r-s}.$$

But the left-hand side is positive because it equals

(33) $$r^k + (s-1)(r^{k-1} + r^{k-2}s + \cdots + rs^{k-2} + s^{k-1}).$$

Thus $r - s > 0$.

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