SEQUENCES OF MATRIX INVERSES FROM PASCAL, CATALAN, AND RELATED CONVOLUTION ARRAYS

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A sequence of sequences S_i arising from the first column of matrix inverses of matrices containing certain columns of Pascal's triangle provided a fruitful study in [1]. Here, we use convolution arrays of the sequences S_i to form a sequence of matrix inverses, leading to inter-relationships between the sequences S_i. The proofs involve generating functions for the columns of infinite matrices, and have diverse applications.

1. SEQUENCES OF MATRIX INVERSES

In this paper, we return to the sequences S_i arising from the first column of P_i^{-1} as in [1]. We form a series of $n \times n$ matrices $P_{i,j}$ by placing every j^{th} column of the convolution triangle for the sequence S_i on and below the main diagonal, and zeroes elsewhere. Then, to relate to the matrix P_i from [1] which was formed by writing the $(i+1)^{st}$ columns of Pascal's triangle on and below the main diagonal, in the new notation, $P_{i-1} = P_{0,i}$, or, every i^{th} column of the convolution array for the sequence $S_0 = \{1, 1, 1, \cdots\}$, which is Pascal's triangle. As a second example, the matrix $P_{1,3}$ would contain every third column of the convolution array for the Catalan sequence S_1 written in triangular form.

We call the inverse of $P_{i,j}$ the matrix $P_{i,j}^{-1}$ and record these inverses in the tables that follow.

Now, let us analyze the results. First, we look at the form of the elements of each matrix inverse, disregarding signs, for $P_{1,j}$. For $j = 1$, the rows of Pascal's triangle appear on and below the main diagonal; these columns are also the columns of the convolution triangle for the sequence $S_{-1} = \{1, 1, 0, 0, 0, \cdots\}$. The column generators, alternating signs included, are $(1-x)^{k-1}$, which are the reciprocals of the column generators for Pascal's triangle, where we do not adjust for the triangular form. For $j = 2$, we have alternate columns of Pascal's triangle, or alternate columns of the convolution triangle for S_0. (In fact, notice that each array contains columns of the convolution array for its left-most column.) For $j = 3$, we have every third column of the convolution triangle for S_1, while $j = 4$ gives fourth columns for S_2.

These results continue for $P_{2,j}^{-1}$ in Table 1.2. When $j = 1$, disregarding the alternating signs of the array, we have every column of the convolution triangle for the sequence $S_{-2} = \{1, 1, -1, 2, -5, 14, -42, \cdots\}$ which contains the Catalan numbers or S_1, taken with alternating signs, following the initial term. If the generating function of S_1 is $C(x)$, then the generating function for S_{-2} is $1/C(x)$. Then, for $j = 2$, we have every second column of the array for S_{-1}; for $j = 3$, every third column of the array for S_0, or, every third column of Pascal's triangle. These results continue, so that when $j = 4$, we have every fourth column of the convolution array for the Catalan numbers, or S_1; $j = 5$, the fifth columns of the array for S_2; $j = 6$, the sixth columns of the array for S_3; and for $j = 7$, the seventh columns of the convolution array for S_4.

Inspecting Table 1.3 for the form of $P_{3,j}^{-1}$ verifies that these results continue. When $j = 1$, every column of the convolution array for the sequence $S_{-3} = \{1, 1, -2, 7, -30, \cdots\}$ appears. Notice that S_{-3} contains the elements of the first convolution of S_2, or of S_2^2, taken with alternating signs and with one additional term preceding the sequence. If the generating function for S_2 is $D(x)$, then the generating function for S_{-3} is $1/D^2(x)$. For $j = 2$, we have every second column of the convolution array for S_{-2}; $j = 3$, every third column of the array for S_{-1}; $j = 4$, every fourth column of the array for S_0; $\cdots$, and for $j = 8$, we have every eighth column of the array for S_4.

Table 1.0
Non-Zero Elements of the Matrices $P_{0,j}^{-1}$ and $P_{0,j}$

	$P_{0,j}^{-1}$						$P_{0,j}$					
	1						1					
	−1	1					1	1				
$j = 1$	1	−2	1				1	2	1			
	−1	3	−3	1			1	3	3	1		
	1	−4	6	−4	1		1	4	6	4	1	
	−1	5	−10	10	−5	1	1	5	10	10	5	1
	1						1					
	−1	1					1	1				
$j = 2$	2	−3	1				1	3	1			
	−5	9	−5	1			1	6	5	1		
	14	−28	20	−7	1		1	10	15	7	1	
	1						1					
	−1	1					1	1				
$j = 3$	3	−4	1				1	4	1			
	−12	18	−7	1			1	10	7	1		
	55	−88	42	−10	1		1	20	28	10	1	
	1						1					
	−1	1					1	1				
$j = 4$	4	−5	1				1	5	1			
	−22	30	−9	1			1	15	9	1		
	140	−200	72	−13	1		1	35	45	13	1	

Table 1.1
Non-Zero Elements of $P_{1,j}^{-1}$ and $P_{1,j}$

	$P_{1,j}^{-1}$						$P_{1,j}$					
	1						1					
	−1	1					1	1				
$j = 1$	0	−2	1				2	2	1			
	0	1	−3	1			5	5	3	1		
	0	0	3	−4	1		14	14	9	4	1	
	1						1					
	−1	1					1	1				
$j = 2$	1	−3	1				2	3	1			
	−1	6	−5	1			5	9	5	1		
	1	−10	15	−7	1		14	28	20	7	1	
	1						1					
	−1	1					1	1				
$j = 3$	2	−4	1				2	4	1			
	−5	14	−7	1			5	14	7	1		
	14	−48	35	−10	1		14	48	35	10	1	
	1						1					
	−1	1					1	1				
$j = 4$	3	−5	1				2	5	1			
	−12	25	−9	1			5	20	9	1		
	55	−130	63	−13	1		14	75	54	13	1	
	−273	700	−408	117	−17	1	42	275	273	104	17	1

Table 1.2

Non-Zero Elements of $P_{2,j}^{-1}$ and $P_{2,j}$

	$P_{2,j}^{-1}$					$P_{2,j}$				
	1					1				
	−1	1				1	1			
j = 1	−1	−2	1			3	2	1		
	−2	−1	−3	1		12	7	3	1	
	−5	−2	0	−4	1	55	30	12	4	1
	1					1				
	−1	1				1	1			
j = 2	0	−3	1			3	3	1		
	0	3	−5	1		12	12	5	1	
	0	−1	10	−7	1	55	55	25	7	1
	1					1				
	−1	1				1	1			
j = 3	1	−4	1			3	4	1		
	−1	10	−7	1		12	18	7	1	
	1	−20	28	−10	1	55	88	42	10	1
	1					1				
	−1	1				1	1			
j = 4	2	−5	1			3	5	1		
	−5	20	−9	1		12	25	9	1	
	14	−75	54	−13	1	55	130	63	13	1
	1					1				
	−1	1				1	1			
j = 5	3	−6	1			3	6	1		
	−12	33	−11	1		12	33	11	1	
	55	−182	88	−16	1	55	182	88	16	1
	1					1				
	−1	1				1	1			
j = 6	4	−7	1			3	7	1		
	−22	49	−13	1		12	42	13	1	
	140	−357	130	−19	1	55	245	117	19	1
	1					1				
	−1	1				1	1			
j = 7	5	−8	1			3	8	1		
	−35	68	−15	1		12	52	15	1	
	285	−606	270	−22	1	55	320	150	22	1

Table 1.3
Non-Zero Elements of $P_{3,j}^{-1}$ and $P_{3,j}$

	$P_{3,j}^{-1}$					$P_{3,j}$				
	1					1				
	−1	1				1	1			
j = 1	−2	−2	1			4	2	1		
	−7	−3	−3	1		22	9	3	1	
	−30	−10	−3	−4	1	140	52	15	4	1
	1					1				
	−1	1				1	1			
j = 2	−1	−3	1			4	3	1		
	−2	0	−5	1		22	15	5	1	
	−5	−1	5	−7	1	140	91	30	7	1
	1					1				
	−1	1				1	1			
j = 3	0	−4	1			4	4	1		
	0	6	−7	1		22	22	7	1	
	0	−4	21	−10	1	140	140	49	10	1
	1					1				
	−1	1				1	1			
j = 4	1	−5	1			4	5	1		
	−1	15	−9	1		22	30	9	1	
	1	−35	45	−13	1	140	200	72	13	1
	1					1				
	−1	1				1	1			
j = 5	2	−6	1			4	6	1		
	−5	27	−11	1		22	39	11	1	
	14	−110	77	−16	1	140	272	99	16	1
	1					1				
	−1	1				1	1			
j = 6	3	−7	1			4	7	1		
	−12	42	−13	1		22	49	13	1	
	55	−245	117	−19	1	140	357	130	19	1
	1					1				
	−1	1				1	1			
j = 7	4	−8	1			4	8	1		
	−22	60	−15	1		22	60	15	1	
	140	−456	165	−22	1	140	456	165	22	1
	1					1				
	−1	1				1	1			
j = 8	5	−9	1			4	9	1		
	−35	81	−17	1		22	72	17	1	
	285	−759	221	−25	1	140	570	204	25	1

To generalize, $P_{i,j}^{-1}$ contains the sequence S_{j-i-1} along its first column and the j^{th} columns of the convolution triangle for the sequence S_{j-i-1}, taken with alternating signs, on and below its main diagonal, with, of course, zeroes everywhere above its main diagonal. The sequences S_i, $i \geqslant 0$, were explored in [1]. The sequences S_{-i}, $i \geqslant 2$, are all related to the sequences S_i by

$$S_{-i} - 1 = S_{i-1}^{i-1}$$

so that, if the initial one is deleted, the sequence S_{-1} is identical to the $(i-2)$nd convolution of the sequence S_{i-1}, $i \geqslant 2$. Also, if the generating function for S_i is $G(x)$, then the generating function for S_{-i-1} is $1/G^i(x)$, $i \geqslant 1$, and S_{-1} has a generating function which is the reciprocal of that for S_0.

There are other patterns which occur for the matrices $P_{i,j}^{-1}$. Except for the alternating signs, $P_{i,j}^{-1}$ is identical to $P_{i,j}$ for $j = 2i + 1$. Furthermore, this property still holds if we form $P_{i,j}^*$ from *any* set of j^{th} columns of the convolution triangle for S_i, $j = 2i + 1$. For example $P_{1,3}^{-1}$ contained the same elements as $P_{1,3}$ except for the alternating signs, where $P_{1,3}$ contained the zeroeth, third, sixth, ⋯, columns of the convolution triangle for S_1 so that its k^{th} column was the $(3k)^{th}$ column of the convolution array. Form $P_{1,3}^*$ to contain every third column of the convolution array for S_1, but beginning from the first convolution, so that the k^{th} column of $P_{1,3}^*$ is the $(3k+1)^{st}$ column of the array, and $P_{1,3}^{*-1}$ has the same elements as $P_{1,3}^*$, taken with alternating signs. Similarly, if we form the matrix $P_{1,3}^{**}$ from the $(3k+2)^{nd}$ columns of the convolution array for S_1, $P_{1,3}^{**-1}$ has the same elements as $P_{1,3}^{**}$ taken with alternating signs. For example, using 5 × 5 matrices,

$$P_{1,3}^{*-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 5 & 5 & 1 & 0 & 0 \\ 14 & 20 & 8 & 1 & 0 \\ 42 & 75 & 44 & 11 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 5 & -5 & 1 & 0 & 0 \\ -14 & 20 & -8 & 1 & 0 \\ 42 & -75 & 44 & -11 & 1 \end{bmatrix}$$

$$P_{1,3}^{**-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 9 & 6 & 1 & 0 & 0 \\ 28 & 27 & 9 & 1 & 0 \\ 90 & 110 & 54 & 12 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 9 & -6 & 1 & 0 & 0 \\ -28 & 27 & -9 & 1 & 0 \\ 90 & -110 & 54 & -12 & 1 \end{bmatrix}$$

Notice that we can consider $n \times n$ submatrices of the infinite matrices of this paper, since for infinite matrices A, B, and C, if we know that $AB = C$ by generating functions, then it must follow that $AB = C$ for $n \times n$ matrices A, B, and C, because each $n \times n$ matrix is the same as the $n \times n$ block in the upper left in the respective infinite matrix. We write the Lemma,

Lemma. Let A be an infinite matrix such that all of its non-zero elements appear on and below its main diagonal, and let $A_{n\times n}$ be the $n \times n$ matrix formed from the upper left corner of A. Let B and C be infinite matrices with $B_{n\times n}$ and $C_{n\times n}$ the $n \times n$ matrices formed from their respective upper left corners. If $AB = C$, then $A_{n\times n}B_{n\times n} = C_{n\times n}$.

Returning for a moment to Tables 1.1, 1.2, and 1.3, notice that the row sums of $P_{1,1}$ are $\{1, 2, 5, 14, 42, \cdots\}$, or S_1^2; the row sums of $P_{2,2}$ are $\{1, 2, 7, 30, 143, \cdots\}$, or S_2^2; and the row sums of $P_{3,3}$ are $\{1, 2, 9, 52, 320, \cdots\}$, or S_3^2. We easily prove that

Theorem. The successive row sums of $P_{i,i}$ are S_i^2.

Proof. Let $S_i(x)$ be the generating function for the sequence S_i. Then the row sums are

$$R(x) = S_i(x) + xS_i^{i+1}(x) + x^2S_i^{2i+1} + \cdots = S_i(x)/[1 - xS_i^i(x)]$$

by summing the infinite geometric series. But, by [1],

$$1 = S_i(x) - xS_i^{i+1}(x),$$

so that $R(x) = S_i^2(x)$ upon simplication.

2. PROOF OF RESULTS AND FURTHER APPLICATIONS

Now, we establish firmly the matrix inverse results of this paper. Let $S_i(x)$ denote the generating function for the sequence S_i, and let $S_k \downarrow S_{k-1}$ mean that S_{k-1} is the solution to

$$S_k(x/s(x)) = S(x),$$

with $S(0) = 1$. It will follow that

$$S_2 \downarrow S_1 \downarrow S_0 \downarrow S_{-1} \downarrow S_{-2} \cdots .$$

Now, it turns out that

$$S_{-1}(x) = \frac{1}{S_0(-x)} = \frac{1}{\frac{1}{1-(-x)}} = 1+x .$$

From this we can show

$$\frac{1}{S_0(-x)} \downarrow \frac{1}{S_1(-x)} \downarrow \frac{1}{S_2(-x)} \cdots$$

as follows:

$$S_1(xS_2(x)) = S_2(x)$$

is trivial, but this continues as

$$\frac{1}{S_1(-x/1/S_2(-x))} = \frac{1}{S_2(-x)},$$

and thus we can generally say

$$\frac{1}{S_m(-x)} \downarrow \frac{1}{S_{m+1}(-x)} .$$

Notice that, if $S_0(x) = 1/(1-x)$, then $S_n(x)$ satisfies

$$\frac{1}{1-xS_n^n(x)} = S_n(x) \quad \text{or} \quad 1 = S_n(x) - xS_n^{n+1}(x).$$

Now, let us look at our general (Pascal) problem. (We denote each matrix by giving successive column generators.) The two infinite matrices

$$(f^m(x), f^{m+k}(x), f^{m+2k}(x), \cdots) \quad \text{and} \quad (A^m(x), A^{m+k}(x), A^{m+2k}(x), \cdots)$$

are matrix inverses if

$$A(x)f(xA^k(x)) = 1 \quad \text{or} \quad f(x/[1/A^k(x)]) = 1/A(x).$$

That is, $1/A(x)$ is k steps down the descending chain of sequences from $f(x)$. Let us examine the two together.

$$(S_2(x), S_2^6(x), S_2^{11}(x), \cdots)^{-1}$$

is given by

$$(A(x), A^6(x), A^{11}(x), \cdots),$$

where

$$S_2(x/1/A^5(x)) = 1/A(x)$$

so we go down five sequences from $S_2(x)$:

$$S_2(x), S_1(x), S_0(x), S_{-1}(x) = 1/S_0(-x), S_{-2}(x) = 1/S_1(-x), S_{-3}(x) = 1/S_2(-x)$$

so that

$$1/A(x) = S_2(-x).$$

This verifies that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 6 & 1 & 0 & 0 \\ 12 & 33 & 11 & 1 & 0 \\ 55 & 182 & 88 & 16 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 3 & -6 & 1 & 0 & 0 \\ -12 & 33 & -11 & 1 & 0 \\ -55 & -182 & 88 & -16 & 1 \end{bmatrix}$$

Lemma. Two infinite matrices

$$(f(x), xf(x)A(x), x^2f(x)A^2(x), x^3f(x)A^3(x), \cdots) \quad \text{and} \quad (g(x), xg(x)B(x), x^2g(x)B^2(x), x^3g(x)B^3(x), \cdots)$$

are inverses if

$$g(x)B(x)A(xB(x))f(xB(x)) = 1 .$$

There are several interesting applications. Consider the central column of Pascal's triangle, $\{1, 2, 6, 20, \cdots\}$ which, upon proper processing, originally gave us the Catalan sequence. Let $f(x)$ be the generating function for the central column of Pascal's triangle, and take

$$f(x) = 1/\sqrt{1-4x}, \quad A(x) = (1-\sqrt{1-4x})/2x, \quad g(x) = 1-2x, \quad B(x) = 1-x,$$

$$A(x(B(x)) = \frac{1-\sqrt{1-4x(1-x)}}{2x(1-x)} = \frac{x}{1-x} = \frac{1}{B(x)}$$

so that $B(x)A(x(B(x)) = 1$. Now

$$f(x(B(x)) = \frac{1}{\sqrt{1-4x(1-x)}} = \frac{1}{1-2x} = \frac{1}{g(x)}$$

so that $g(x)f(xB(x)) = 1$ also. This matrix uses elements from the central column of Pascal's triangle and the columns parallel to it, and has inverse whose columns have the coefficients from the generating functions. For example, for the 5×5 case,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 10 & 10 & 4 & 1 & 0 \\ 20 & 35 & 56 & 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 2 & -4 & 1 & 0 \\ 0 & 0 & 5 & -5 & 1 \end{pmatrix}$$

If we now go to the m^{th} columns, then

$$g(x)B^m(x)A^m(x(B(x))f(xB(x)) = 1$$

so it seems to naturally break into two separate parts:

$$(1) \qquad B(x)A(x(B(x)) = 1$$

$$(2) \qquad g(x)f(xB(x)) = 1,$$

where we already know how to solve (1), but (2) is something new when combined with (1) since the above has to hold for all $m \geq 0$.

Let us consider $S_2^* = \{1, 3, 15, 84, \cdots\}$, the diagonal of Pascal's triangle, which, upon proper processing, lead to our sequence $S_2 = \{1, 1, 3, 12, 55, 273, \cdots\}$. Let $S_2^*(x)$ be the generating function for S_2^*, and take

$$f(x) = S_2^*(x), \qquad A(x) = S_2(x),$$

and let $S_1(x) = C(x)$ be the generating function for the sequence $S_1 = \{1, 1, 2, 5, 14, 70, \cdots\}$, the Catalan sequence. Then,

$$\begin{aligned} g(x) &= 1 - 3xC(x) & B(x) &= 1 - xC(x) \\ &= \frac{3}{C(x)} - 2 & &= \frac{1}{C(x)} \\ &= \frac{3}{S_1(x)} - 2 & &= \frac{1}{S_1(x)} \end{aligned}$$

Now, let $S_3^* = \{1, 4, 28, 220, \cdots\}$, the diagonal of Pascal's triangle which led to the sequence S_3. Here, we use

$$1 = S_3(x) - xS_3^3(x)$$

and we can write

$$f(x) = S_3^*(x), \quad A(x) = S_3(x), \quad B(x) = 1/S_3(x), \quad \text{and} \quad g(x) = 1 - 4xS_3^2(x) = 4/S_3(x) - 3.$$

Generally speaking, we take

$$f(x) = S_k^*(x), \quad A(x) = S_k(x), \quad B(x) = 1/S_k(x), \quad g(x) = 1-(k+1)xS_k^{k-1}(x) = \frac{k+1}{S_k(x)} - k.$$

Lemma. The two infinite matrices

$$(f(x)A^m(x), xf(x)A^{m+k}(x), x^2f(x)A^{m+2k}(x), \cdots) \quad \text{and} \quad (g(xB^m(x), xg(x)B^{m+k}(x), x^2g(x)B^{m+2k}(x), \cdots)$$

are inverses if $f(xB^k(x))g(x) = 1$ and $A(xB^k(x))B(x) = 1$, simultaneously.

The Lemma is the same as considering the two infinite matrices

$$(F(x), xF(x)A^k(x), x^2F(x)A^{2k}(x), \cdots) \quad \text{and} \quad (G(x), xG(x)B^k(x), x^2G(x)B^{2k}(x), \cdots),$$

where

$$F(x) = f(x)A^m(x); \quad G(x) = g(x)B^m(x); \quad A^k(xB^k(x))f(xB^k(x))A^m(xB^k(x))B^m(x)g(x) = 1$$

or

$$[A(xB^k(x))B(x)]^m = 1 \quad \text{and} \quad f(xB^k(x))g(x) = 1, \quad A(0) = B(0) = 1.$$

With application to the sequences S_i of this paper, we can take

$$f(x) = D_0(x), \quad A(x) = S_k(x), \quad g(x) = 1-(k+1)S_k^{k-1}(x), \quad \text{and} \quad B(x) - 1/S_{k-1}(x).$$

The above lemma can also be illustrated by taking

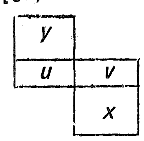
$$f(x) = 1/(1-x), \quad A(x) = (1+x)/(1-x), \quad g(x) = (3+x-\sqrt{1+6x+x^2})/2,$$

and

$$B(x) = [-(1+x)+\sqrt{1+6x+x^2}]/2x.$$

This arises from the triangular matrix (from a paper by Alladi [6])

1				
1	1			
1	3	1		
1	5	5	1	
1	7	13	7	1
………				



$x = u+v+y$

where the column generators are successively given by

$$\frac{1}{1-x}, \quad \frac{x(1+x)}{(1-x)^2}, \quad \cdots, \quad \frac{x^n(1+x)^n}{(1-x)^{n+1}}, \quad \cdots$$

The lemmas of this section also apply to some other interesting sequences. Suppose we take the sequence $\{1, 1, 2, 4, 8, 16, \cdots\}$ which is generated by

$$f(x) = \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x} = 1 + \sum_{n=0}^{\infty} 2^n x^n .$$

Let $Hf(x) = S(x)$, where $S(0) = 1$ and $S(x)$ satisfies $f(xS(x)) = S(x)$. Then $H^2 f(x) = S(x)$ means that $f(xS^2(x)) = S(x)$, $S(0) = 1$.

$$H\left(\frac{1-x}{1-2x}\right) = g(x)$$

which is the generating function for $\{1, 1, 3, 11, 45, 197, 903, \cdots, \}$. (See Riordan [2], p. 168), while

$$H^2\left(\frac{1-x}{1-2x}\right) = H(g(x)) = h(x),$$

which is the generating function for the sequence $\{1. 1, 4, 21, 126, 818, 5594, \cdots\}$ given by Carlitz [4]. There is another sequence from the same article by Carlitz, but first we note $B = \{1, 1, 3, 11, 45, \cdots, b_n, \cdots\}$ obeys

$$(n+1)b_n - 3(2n-1)b_{n-1} + (n-2)b_{n-2} = 0.$$

We solve the quadratic

$$2xS^2 - (x+1)S + 1 = 0$$

$$S(x) = \frac{1 + x \pm \sqrt{1 + 2x + x^2 - 8x}}{4x}$$

$$S(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x} = \sum_{n=0}^{\infty} b_n x^n \ .$$

From this, we should be able to establish the recurrence. We also note that, where $C(x) = (1 - \sqrt{1-4x}\,)/2x$ is the generator for the Catalan sequence,

$$\frac{1}{1+x}\, C\left(\frac{2x}{(1+x)^2}\right) = S(x)$$

which comes from Riordan [2], p. 168.

There is another application. Let $f(x)$ generate the odd numbers. Then the solution to $f(x(S(x)) = S(x)$, $S(0) = 1$, is the sequence $\{1, 3, 14, 79, 494, 3294, \cdots\}$ given by Carlitz [4], which has generating functions

$$\frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 + \cdots$$

$$\frac{1 + xS(x)}{(1 - xS(x))^2} = S(x)$$

$$1 + xS(x) = S(x) - 2xS^2(x) + x^2S^3(x)$$

$$0 = x^2S^3(x) - 2xS^2(x) + (1-x)S(x) - 1,$$

where $S(x)$ generates $\{1, 3, 14, 79, 494, 3294, \cdots\}$. As verification,

$$S = \{1, 3, 14, 79, 494, 3294, \cdots\}, \quad S^2 = \{1, 6, 37, 242, 1658, \cdots\}$$

$$S^3 = \{1, 9, 69, 516, \cdots\}, \quad -1S^0 = \{-1, 0, 0, 0, 0, 0, \cdots\}$$

$$S = \{1, 3, 14, 79, 494, 3294, \cdots\}, \quad -xS = \{0, -1, -3, -14, -79, -494, \cdots\}$$

$$-2xS^2 = \{0, -2, -12, -74, -484, -3316, \cdots\}, \quad x^2S^3 = \{0, 0, 1, 9, 69, 516, \cdots\}$$

with all vertical sums equalling zero.

With the methods of this section, it is easily shown that, if $P^*_{i,j}$ is the matrix formed by moving each column of $P_{i,j}$ up to form a rectangular array, $P^{-1}_{i,j}P^*_{i,j}$ contains Pascal's triangle written in rectangular form, which is such a prolific result that it is the content of another paper [3]. Paul Bruckman has proved the matrix theorems used in this section in [5]. See [7] also.

REFERENCES

1. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Catalan and Related Sequences Arising from Inverses of Pascal's Triangle Matrices," *The Fibonacci Quarterly,* Vol. 14, No. 4 (Dec. 1976), to appear.
2. John Riordan, *Combinatorial Analysis,* John Wiley and Sons, 1968.
3. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Pascal, Catalan, and General Sequence Convolution Arrays in a Matrix," *The Fibonacci Quarterly,* Vol. 14, No. 2 (April, 1976), pp. 135–142.
4. L. Carlitz, "Enumeration of Two-Line Arrays," *The Fibonacci Quarterly,* Vol. 11, No. 2 (April, 1973), pp. 113–130.
5. Paul Bruckman, Private Communication
6. Krishnaswami Alladi, Private Communication.
7. V. E. Hoggatt, Jr., and Paul S. Bruckman, "The H-Convolution Transform," *The Fibonacci Quarterly,* Vol. 13, No. 4 (December 1975), pp. 357–367.

NOTE: In the paper, "Pascal, Catalan, and General Sequence Convolution Arrays in a Matrix," the publication dates of references 1 and 2 were inadvertently interchanged. Reference 4 should have read: Michael Rondeau, "The Generating Functions for the Vertical Columns of (N + 1)-Nomial Triangles," unpublished Master's Thesis, San Jose State University, December, 1973.

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