SUMS OF COMBINATION PRODUCTS

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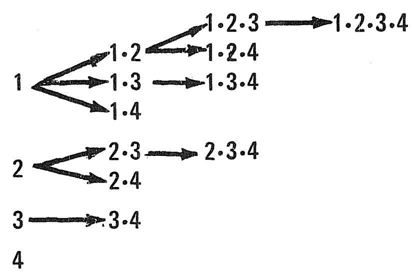
INTRODUCTION

The combinations of the integers 1, 2, 3, 4 can be represented by the following diagram:



We will be interested in developing methods for evaluating sums of the form

$$1\cdot2+1\cdot3+1\cdot4+2\cdot3+2\cdot4+3\cdot4 \qquad \text{and} \qquad 1\cdot2\cdot3+1\cdot2\cdot4+1\cdot3\cdot4+2\cdot3\cdot4\,.$$

We let

$$\sum_{\substack{x_1<\cdots<x_r\\ M_n}} (x_1\cdot x_2\cdot\cdots\cdot x_r)$$

denote the sum of all products of the form $x_1\cdot x_2\cdot\cdots\cdot x_r$, where

$$x_1<x_2<\cdots<x_r;\quad x_1,x_2,\cdots,x_r\in\{1,2,\cdots,n\},\quad \text{and}\quad n\geq r\geq 2.$$

For example,

$$\sum_{\substack{x_1<x_2\\ M_4}} x_1x_2 = 1\cdot2+1\cdot3+1\cdot4+2\cdot3+2\cdot4+3\cdot4 \quad\text{and}\quad \sum_{\substack{x_1<x_2\\ M_3}} x_1x_2 = 1\cdot2+1\cdot3+2\cdot3\,.$$

We define

$$A_r^n = \sum_{\substack{x_1<\cdots<x_r\\ M_n}} (x_1\cdot x_2\cdot\cdots\cdot x_r),\qquad r\geq 2,\qquad\text{and}\qquad A_1^n=\sum_{i=1}^n i\,.$$

In this paper we develop formulas for A_2^n, A_3^n, A_4^n. We also provide a general approach for finding A_r^n when $n\geq r\geq 5$.

A. We now develop a formula for A_2^n. Consider

$$\left(\sum_{i=1}^n i\right)\left(\sum_{j=1}^n j\right)=\sum_{i=1}^n\sum_{j=1}^n ij=\sum_{i\neq j} ij+\sum_{i=1}^n i^2\,.$$

Thus,

(1) $$\sum_{i \neq j} ij = \left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i^2 .$$

Now,

$$\sum_{i \neq j} ij = 2 \sum_{\substack{i<j \\ M_n}} ij .$$

Thus,

$$2 \sum_{\substack{i<j \\ M_n}} ij = \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6} .$$

Thus, we have

Theorem 1. Say $n \geqslant 2$. Then

$$2 \sum_{\substack{i<j \\ M_n}} ij = \left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i^2 = \frac{3(n^4 - n^2) + 2(n^3 - n)}{4(3)} .$$

For example, with $n = 3$,

$$2(1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3) = \frac{3(3^4 - 3^2) + 2(3^3 - 3)}{4(3)} .$$

We could also find

$$\sum_{\substack{i<j \\ M_n}} ij$$

by using the method of undetermined coefficients. We begin by assuming that

$$\sum_{\substack{x_1 < \cdots < x_r \\ M_n}} (x_1 \cdot \cdots \cdot x_r)$$

is a polynomial of degree $2r$ in n (we assume that the coefficient of n^0 is zero):

$$2 \sum_{\substack{i<j \\ M_n}} ij = An^4 + Bn^3 + Cn^2 + Dn .$$

Now,

$$\sum_{\substack{i<j \\ M_2}} ij = 1 \cdot 2 = 2, \quad \sum_{\substack{i<j \\ M_3}} ij = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11 ,$$

$$\sum_{\substack{i<j \\ M_4}} ij = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35 ,$$

$$\sum_{\substack{i<j \\ M_5}} ij = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 1 \cdot 5 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 + 3 \cdot 4 + 3 \cdot 5 + 4 \cdot 5 = 85 .$$

Thus,

$$2(2) = A\cdot 2^4 + B\cdot 2^3 + C\cdot 2^2 + D\cdot 2, \quad 2(11) = A\cdot 3^4 + B\cdot 3^3 + C\cdot 3^2 + D\cdot 3,$$
$$2(35) = A\cdot 4^4 + B\cdot 4^3 + C\cdot 4^2 + D\cdot 4, \quad 2(85) = A\cdot 5^4 + B\cdot 5^3 + C\cdot 5^2 + D\cdot 5.$$

Solving this system for A,B,C,D should provide the required answer. Generalizing Theorem 1, we have

Theorem 2. Say $a_i, a_j \in \{a_1, a_2, \cdots, a_n\}$ and $n \geq 2$. Then

$$2\sum_{i<j} a_i a_j = \left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2 .$$

For example, letting $a_i = i^2$,

$$2\sum_{i<j} (ij)^2 = \left(\sum_{i=1}^{n} i^2\right)^2 - \sum_{i=1}^{n} i^4 .$$

Similarly, letting $a_i = 1/i$,

$$2\sum_{i<j} \frac{1}{ij} = \left(\sum_{i=1}^{n} \frac{1}{i}\right)^2 - \sum_{i=1}^{n} \frac{1}{i^2} .$$

For example,

$$2\left(\frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3}\right) = \left(1 + \frac{1}{2} + \frac{1}{3}\right)^2 - \left(1 + \frac{1}{4} + \frac{1}{9}\right).$$

Now, say $x^3 + Bx^2 + Cx + D = 0$. Then, by Theorem 2, letting a_i equal the i^{th} root of the above equation,

$$2C = B^2 - (a_1^2 + a_2^2 + a_3^2).$$

Say $B = C = 0$. Then $a_1^2 + a_2^2 + a_3^2 = 0$. Thus, we have

Theorem 3. Say $r_1, r_2, \cdots, r_n$ are the roots of $x^n = -D$, and $n \geq 3$. Then $r_1^2 + r_2^2 + \cdots + r_n^2 = 0$.

B. We now develop a formula for A_3^n. Consider:

$$\left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j\right)\left(\sum_{k=1}^{n} k\right) = \left(\sum_{i=1}^{n} i\right)^3 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk.$$

We consider

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk$$

to be a sum of products having three factors. Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk = \sum_{\substack{\text{all factors}\\ \text{equal}}} ijk + \sum_{\substack{\text{all factors}\\ \text{different}}} ijk + \sum_{\substack{\text{two factors}\\ \text{equal}}} ijk .$$

Now, if the product $1\cdot 2\cdot 4$ appears in the sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk,$$

the following products also appear:

$$1\cdot 4\cdot 2, \quad 2\cdot 1\cdot 4, \quad 2\cdot 4\cdot 1, \quad 4\cdot 1\cdot 2, \quad 4\cdot 2\cdot 1 .$$

These may be considered as rearrangements of $1\cdot 2\cdot 4$. We note that the number of permutations of three distinct objects taken three at a time is six.

If the product $1\cdot 1\cdot 4$ appears in the sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk,$$

the following products also appear:

$$1\cdot 4\cdot 1, \qquad 4\cdot 1\cdot 1.$$

We note that the number of permutations of three objects, two of which are of one kind, is three. Thus,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} ijk = \sum_{i=1}^{n} i^3 + 6 \sum_{\substack{i<j<k\\ M_n}} ijk + \left(3\sum_{i=1}^{n} 1^2 i - 3\cdot 1^3\right) + \left(3\sum_{i=1}^{n} 2^2 i - 3\cdot 2^3\right)$$

$$+ \cdots + \left(3\sum_{i=1}^{n} n^2 i - 3n^3\right) = \sum_{i=1}^{n} i^3 + 6\sum_{\substack{i<j<k\\ M_n}} ijk + 3\sum_{i=1}^{n} i(1^2 + 2^2 + \cdots + n^2) - 3(1^3 + 2^3 + \cdots + n^3)$$

$$= \sum_{i=1}^{n} i^3 + 6\sum_{\substack{i<j<k\\ M_n}} ijk + 3\left(\sum_{i=1}^{n} i\right)\left(\sum_{i=1}^{n} i^2\right) - 3\sum_{i=1}^{n} i^3.$$

Thus, we have

Theorem 4. Say $n \geqslant 3$. Then

$$6\sum_{\substack{i<j<k\\ M_n}} ijk = \left(\sum_{i=1}^{n} i\right)^3 + 2\sum_{i=1}^{n} i^3 - 3\left(\sum_{i=1}^{n} i^2\right)\left(\sum_{i=1}^{n} i\right) = \frac{n^6}{8} - \frac{n^5}{8} - \frac{3n^4}{8} + \frac{n^3}{8} + \frac{n^2}{4}.$$

For example, with $n = 4$,

$$6(1\cdot 2\cdot 3 + 1\cdot 2\cdot 4 + 1\cdot 3\cdot 4 + 2\cdot 3\cdot 4) = \left(\sum_{i=1}^{4} i\right)^3 + 2\sum_{i=1}^{4} i^3 - \left(\sum_{i=1}^{4} i^2\right)\left(\sum_{i=1}^{4} i\right).$$

We now give an alternate derivation of the formula for $\sum_{\substack{i<j<k\\ M_n}} ijk$

Consider: $3(1\cdot 2) + 4(1\cdot 2 + 1\cdot 3 + 2\cdot 3) + 5(1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 3 + 2\cdot 4 + 3\cdot 4) = (1\cdot 2\cdot 3) + (1\cdot 2\cdot 4 + 1\cdot 3\cdot 4 + 2\cdot 3\cdot 4) + (1\cdot 2\cdot 5 + 1\cdot 3\cdot 5 + 1\cdot 4\cdot 5 + 2\cdot 3\cdot 5 + 2\cdot 4\cdot 5 + 3\cdot 4\cdot 5)$. This suggests that

$$\sum_{\substack{i<j<k\\ M_n}} ijk = 3\sum_{\substack{i<j\\ M_2}} ij + 4\sum_{\substack{i<j\\ M_3}} ij + \cdots + n\sum_{\substack{i<j\\ M_{n-1}}} ij.$$

Thus, we conjecture,

$$\sum_{\substack{i<j<k\\ M_n}} ijk = \sum_{w=2}^{n-1} (w+1)\sum_{\substack{i<j\\ M_w}} ij. \tag{2}$$

Thus, by Theorem 1, we conjecture

$$\sum_{\substack{i<j<k\\ M_n}} ijk = \sum_{w=2}^{n-1} (w+1)\left[\frac{3(w^4 - w^2) + 2(w^3 - w)}{24}\right]$$

and we have

Theorem 5. Say $n \geqslant 3$. Then

$$24 \sum_{\substack{i<j<k \\ M_n}} ijk = \sum_{i=1}^{n} (3i^5 + 5i^4 - i^3 - 5i^2 - 2i) - 3n^5 - 5n^4 + n^3 + 5n^2 + 2n .$$

We can prove Theorem 5 by using Theorem 4 and the following formulas:

$$2\sum_{i=1}^{n} i = n^2 + n, \quad 3\sum_{i=1}^{n} i^2 = n^3 + \frac{3n^2}{2} + \frac{1n}{2}, \quad 4\sum_{i=1}^{n} i^3 = n^4 + 2n^3 + n^2,$$

$$5\sum_{i=1}^{n} i^4 = n^5 + \frac{5n^4}{2} + \frac{5n^3}{3} - \frac{1n}{6}, \qquad 6\sum_{i=1}^{n} i^5 = n^6 + 3n^5 + \frac{5n^4}{2} - \frac{1n^2}{2} .$$

C. We now develop a formula for A_4^n. Consider:

$$4(1\cdot2\cdot3) + 5(1\cdot2\cdot3 + 1\cdot2\cdot4 + 1\cdot3\cdot4 + 2\cdot3\cdot4) = (1\cdot2\cdot3\cdot4) + (1\cdot2\cdot3\cdot5 + 1\cdot2\cdot4\cdot5 + 1\cdot3\cdot4\cdot5 + 2\cdot3\cdot4\cdot5) .$$

This suggests that

$$\sum_{\substack{i<j<k<\ell \\ M_n}} ijk\ell = 4 \sum_{\substack{i<j<k \\ M_3}} ijk + 5 \sum_{\substack{i<j<k \\ M_4}} ijk + \cdots + n \sum_{\substack{i<j<k \\ M_{n-1}}} ijk .$$

Thus, we conjecture,

$$\sum_{\substack{i<j<k<\ell \\ M_n}} ijk\ell = \sum_{w=3}^{n-1} (w+1) \sum_{\substack{i<j<k \\ M_w}} ijk . \tag{3}$$

Thus, by Theorem 4, we conjecture,

$$\sum_{\substack{i<j<k<\ell \\ M_n}} ijk\ell = \sum_{w=3}^{n-1} \frac{(w+1)}{24} \left(\frac{w^6}{2} - \frac{w^5}{2} - \frac{3w^4}{2} + \frac{w^3}{2} + w^2 \right)$$

and we have

Conjecture 1. Say $n \geq 4$. Then

$$24 \sum_{\substack{i<j<k<\ell \\ M_n}} ijk\ell = \sum_{i=1}^{n} \left(\frac{1i^7}{2} - 2i^5 - i^4 + \frac{3i^3}{2} + i^2 \right) - \frac{n^7}{2} + 2n^5 + n^4 - \frac{3n^3}{2} - n^2 .$$

Comparing (2) and (3) we have

Conjecture 2. Say $n \geq r \geq 3$. Then

$$\sum_{\substack{x_1<\cdots<x_r \\ M_n}} \prod_{i=1}^{r} x_i = \sum_{w=r-1}^{n-1} (w+1) \sum_{\substack{x_1<\cdots<x_{r-1} \\ M_w}} \prod_{i=1}^{r-1} x_i .$$

Thus, we have

Conjecture 3. Conjecture 2 and Theorem 1 provide a recursive method for determining $A_3^n, A_4^n, A_5^n, \cdots$.

D. *Theorem 6.* Say $n \geq 2$. Then

$$(n-1)! = n^{n-1} + \sum_{i=1}^{n-1} (-1)^i A_i^{n-1} n^{n-(i+1)} .$$

Proof.

$$(n-1)! = (n-1)(n-2)\cdots[n-(n-1)] = n^{n-1} + (-1)^1 A_1^{n-1} n^{n-2} + (-1)^2 A_2^{n-1} n^{n-3} + (-1)^3 A_3^{n-1} n^{n-4} + \cdots + (-1)^{n-1} A_{n-1}^{n-1} n^{n-n}.$$

E. *Theorem 7.* The A_i^n can be solved for by Cramer's rule. Also,

$$\sum_{i=1}^{n} A_i^n = (n+1)! - 1.$$

Proof. Let $f(x) = (x+1)(x+2)\cdots(x+n) = (x+n)!/x!$. Then $f(x) = x^n + A_1^n x^{n-1} + A_2^n x^{n-2} + \cdots + A_{n-1}^n x + A_n^n$. Thus,

$$A_1^n 1^{n-1} + A_2^n 1^{n-2} + \cdots + A_{n-1}^n 1^1 + A_n^n = f(1) - 1^n$$
$$A_1^n 2^{n-1} + A_2^n 2^{n-2} + \cdots + A_{n-1}^n 2^1 + A_n^n = f(2) - 2^n$$
$$\vdots$$
$$A_1^n n^{n-1} + A_2^n n^{n-2} + \cdots + A_{n-1}^n n^1 + A_n^n = f(n) - n^n,$$

where the A_i^n can be solved for by Cramer's rule.

F. *Theorem 8.* Say $n \geq r \geq 1$ and $f(x) = (x+n)!/x!$. Then

$$A_r^n = \frac{f^{[n-r]}(0)}{(n-r)!},$$

where $f^{[n-r]}(0)$ denotes the $n-r$ derivative evaluated at zero.

Proof. Say $f(x) = (x+1)(x+2)\cdots(x+n) = (x+n)!/x!$. Then $f(x) = x^n + A_1^n x^{n-1} + \cdots + A_{n-1}^n x + A_n^n$. Now $f(x)$ is a polynomial of degree n. Thus, by Taylor's formula,

$$f(x) = f(0) + f^{[1]}(0)x + \frac{f^{[2]}(0)x^2}{2!} + \cdots + \frac{f^{[n]}(0)x^n}{n!}.$$

Thus, comparing the coefficients of the above two equations, the theorem is proved.

G. A Curiosity. Let

(4)
$$T_Q = -\sum_{x_1=1}^{Q} x_1 + \sum_{x_1=2}^{Q} x_1 \sum_{x_2=1}^{x_1-1} x_2 - \sum_{x_1=3}^{Q} x_1 \sum_{x_2=2}^{x_1-1} x_2 \sum_{x_3=1}^{x_2-1} x_3 + \sum_{x_1=4}^{Q} x_1 \sum_{x_2=3}^{x_1-1} x_2 \sum_{x_3=2}^{x_2-1} x_3 \sum_{x_4=1}^{x_3-1} x_4 - \cdots + w(Q,Q),$$

where

$$w(v,Q) = (-1)^v \sum_{x_1=v}^{Q} x_1 \left[\sum_{x_2=v-1}^{x_1-1} x_2 \sum_{x_3=v-2}^{x_2-1} x_3 \cdots \sum_{x_v=1}^{x_{v-1}-1} x_v \right].$$

$$\therefore\ T_Q = -\sum_{x_1=1}^{Q} x_1 + \sum_{v=2}^{Q} w(v,Q).$$

Thus,

(5)
$$T_1 = -[1]$$

(6)
$$T_2 = -[1+2] + [2(1)]$$

(7)
$$T_3 = -[1+2+3] + [2(1) + 3(1+2)] - \left\{ 3[2(1)] \right\} = -[1+2+3] + [(2\cdot1) + (3\cdot1) + (3\cdot2)] - [(3\cdot2\cdot1)].$$

This suggests

Conjecture 4.

$$A_2^n = \sum_{x_1=2}^{n} x_1 \sum_{x_2=1}^{x_1-1} x_2, \quad A_3^n = \sum_{x_1=3}^{n} x_1 \sum_{x_2=2}^{x_1-1} x_2 \sum_{x_3=1}^{x_2-1} x_3$$

$$\cdots A_{n-1}^n = \sum_{x_1=n-1}^{n} x_1 \sum_{x_2=n-2}^{x_1-1} x_2 \sum_{x_3=n-3}^{x_2-1} x_3 \cdot \cdots \cdot \sum_{x_{n-1}=1}^{x_{n-2}-1} x_{n-1}$$

and

Conjecture 5.

$$T_n = \sum_{i=1}^{n} (-1)^i A_i^n .$$

We note that from Conjecture 4,

$$A_2^n \overset{?}{=} \sum_{x_1=2}^{n} x_1 \sum_{x_2=1}^{x_1-1} x_2 = \sum_{i=2}^{n} \sum_{j=1}^{i-1} ij = \sum_{i=2}^{n} \frac{i(i-1)i}{2} = \frac{1}{2} \sum_{i=2}^{n} (i^3 - i^2)$$
$$= \frac{1}{2} \left\{ \left[\frac{n(n+1)}{2} \right]^2 - \frac{n(n+1)(2n+1)}{6} \right\}$$

which agrees with Theorem 1.

Similarly,

$$A_3^n \overset{?}{=} \sum_{x_1=3}^{n} x_1 \sum_{x_2=2}^{x_1-1} x_2 \sum_{x_3=1}^{x_2-1} x_3 = \sum_{i=3}^{n} \sum_{j=2}^{i-1} \sum_{k=1}^{j-1} ijk .$$

We note that $T_3 = T_2 - 3 + 3[1+2] - 3[2(1)] = T_2 - 3 - 3T_2$ and $T_4 = T_3 - 4 + 4[1+2+3] - 4[2(1) + 3(1+2)] + 4 \left\{ 3[2(1)] \right\} = T_3 - 4 - 4T_3$. Thus, $T_3 = -2T_2 - 3$ and $T_4 = -3T_3 - 4$. This suggests

Theorem 9. Say $Q \geqslant 1$. Then

(8) $$T_Q = -(Q-1)T_{Q-1} - Q.$$

We leave the proof to the reader.

We might hope that the T_Q represent a new species of number. Let's see; i.e., from (5), (6), (7) we have

$$T_1 = -1, \qquad T_2 = -3+2 = -1, \qquad T_3 = -6+11-6 = -1.$$

This suggests

Theorem 10. Say $Q \geqslant 1$. Then $T_Q = -1$.

Proof (induction). By (5) we know that $T_1 = -1$. Say k is a fixed integer greater than or equal to two and $T_{k-1} = -1$. Then, by (8), $T_k = -1$ and the theorem is proved.

Hence, from Conjecture 5 and the above theorem, we have

Conjecture 6.

$$\sum_{i=1}^{n} (-1)^i A_i^n = -1 .$$

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