A FORMULA FOR FIBONACCI NUMBERS FROM A NEW APPROACH TO GENERALIZED FIBONACCI NUMBERS

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INTRODUCTION

Ever since the establishment of the Fibonacci Association and its main publication, *The Fibonacci Quarterly,* under the devoted guidance of its founder, the Fibonacci master, Verner E. Hoggatt, Jr., [3] of San Jose State University, California, the study of the Fibonacci sequence

(0.1) $$F_1 = F_2 = 1; \qquad F_{n+2} = F_n + F_{n+1}; \qquad (n = 1, 2, \cdots)$$

has seen a new and rapid development in the last two decades. The impressive list of brilliant mathematicians who have contributed is too long to be mentioned here. But the author thinks that the time is ripe for some kind of a Dickson-survey of all the splendid results in the Fibonacci Wonderland which has fascinated mathematicians for the last 775 years, since the son of Bonacci wrote his *Liber abaci* in 1202.

Together with the study of the original Fibonacci sequence (0.1) went the generalization of these sequences. This was a result of pure mathematical curiosity and speculative creativity, without any application to the freightening population explosion of rabbits. This generalization could lead into various directions. First—the initial values of F_1, F_2 in (0.1) could be arbitrarily chosen, and this gave birth to the Lucas numbers, in addition to many other step-children. The most reckless, most general generalization, taking us to dimensions beyond the imagination or needs of Leonardo da Pisa, would be the following: let

(0.2) $$\begin{cases} F_j = a_j & (j = 1, \cdots, n) \\ F_{n+v} = \sum_{i=0}^{n-1} b_i F_{v+i}, & (v = 1, 2, \cdots) \\ a_j, b_i \in Z; & a_j, b_i \text{ fixed.} \end{cases}$$

Of course, it is possible to drive this inconsiderateness still further and choose a_j, b_i from C. But one should make a halt somewhere. In a previous paper the author [1,a], and in a joint paper Hasse and the author [1,b] have investigated the most simple case of the general generalization of the Fibonacci numbers, viz.

(0.3) $$\begin{cases} F_1 = 1, \quad F_i = 0 & (i = 2, \cdots, n), \\ F_{n+v} = \sum_{i=0}^{n-1} F_{v+i} & (v = 1, 2, \cdots). \end{cases}$$

The author succeeded to calculate F_{n+v} in a comparatively simple explicit formula. In principle, this is possible also for F_{n+v} from (0.2), by means of Euler's generating functions. The author applied, for the calculation of F_{n+v} from (0.3), the Jacobi-Perron algorithm [1,c], which led him to suggest that the sequence of the original Fibonacci numbers should actually be defined by

(0.4) $$F_{-1} = 1; \quad F_0 = 0; \quad F_{n+2} = F_n + F_{n+1} \quad (n = -1, 0, 1, \cdots).$$

While trying to generalize the original Fibonacci number to higher dimensions, one is immediately exposed to the danger of losing the royal property of the original Fibonacci numbers, viz., $F_m | F_{mk}$. This damage has not yet been repaired for Fibonacci numbers of dimension $n > 2$, and the author conjectures that this will remain a utopia.

In this context the question arises: what is the natural generalization of the Fibonacci numbers, if any? For this purpose, one should look into another direction than (0.2).

As is known, the generating polynomial for the original Fibonacci numbers is

$$\left\{\begin{array}{ll} P(x) = x^2 - x - 1; & P(\alpha) = P(\beta) = 0 \\ \alpha = \dfrac{1+\sqrt{5}}{2}, & \beta = \dfrac{1-\sqrt{5}}{2} \end{array}\right. \tag{0.5}$$

from which, by easy calculations, the two well known formulas are derived:

$$\left\{\begin{array}{l} F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta} = \sum\limits_{i=0} \binom{n-1-i}{i} ; \\ n = 1, 2, \cdots; \binom{0}{0} \overset{\text{def}}{=} 1. \end{array}\right. \tag{0.6}$$

As is seen from (0.5), α and β are units in $Q(\sqrt{5})$. Generalizing (0.5), and demanding that the two roots of the new polynomial be units, one would suggest that the natural generalization of the generating polynomial would be

$$P(x) = x^2 - ax - 1, \quad a \in N. \tag{0.7}$$

By a technique which will be developed in the next chapter, one obtains generalized Fibonacci numbers (of dimension two) $F_{a,n}$ of the form

$$\left\{\begin{array}{l} F_{a,n} = \sum\limits_{i=0} \binom{n-1-i}{i} a^{n-1-2i}, \quad (n = 1, 2, \cdots) \\ F_{a,n+2} = F_{a,n} + aF_{a,n+1} . \end{array}\right. \tag{0.8}$$

For $a = 1$, (0.8) become the original Fibonacci numbers, as should be. But, alas, we had hoped to arrive at a new formula for F_n. So the generalization (0.7) does not supply the natural generalization for the original Fibonacci numbers, and new horizons must be searched.

1. THE NEW APPROACH

In two previous papers [1,d), e)] the author has established a few new combinatorial identities by means of a new technique. These identities are of a quite complicated nature, and only a combinatorial master like Leonard Carlitz [2] could have succeeded to prove them by elementary tools. The basic ideas of this new technique, together with a few illustrations, will soon appear in a paper; an abstract [1,f)] of this paper has been published. The author doubts not that mathematicians, once they have become familiar with this technique, will come up with a treasure of new and interesting combinatorial identities which could probably not be proved with elementary means.

A word about its contents. Since the new technique is based on the knowledge of one or more independent units in an algebraic number field of any degree $n \geqslant 2$, these units must, of course, be explicitly stated. Now, there are many elaborate methods to find the basis (the maximal set of fundamental units) of the multiplicative group of units in a numeric, given algebraic number field $Q(w)$,

$$\left\{\begin{array}{l} w^n + k_1 w^{n-1} + \cdots + k_{n-1} w + k_n = 0, \\ k_i \in Z, \quad k_i \text{ fixed} \quad (i = 1, \cdots, n). \end{array}\right. \tag{1.1}$$

The situation is entirely different, if the $k_1, \cdots, k_n$ from (1.1) are any free parameters. In this case we speak about $Q(w) = Q(w; k_1, \cdots, k_n)$ as a functional algebraic number field. In this case it seems almost impossible to state one or more independent units in $Q(w; k_1, \cdots, k_n)$ explicitly (they must not be fundamental). We do dare to think that the author, and in a few joint papers the author and Hasse [1,c] were the first pioneers to state explicitly units in functional algebraic number fields of any degree $n \geqslant 2$. Of course, if $k_n = \pm 1$ in (1.1), then w is always a unit. This led to the original generating polynomial (0.5) for the original Fibonacci numbers, and its generalization (0.7), which, as we have seen led to essentially nothing new, but an identity of the numbers $F_{a,n}$ from (0.8) and a polynomial in $\frac{1}{2}(1 \pm \sqrt{a^2+4})$. For generalized Fibonacci numbers of dimension $n \geq 3$, it will be definitely worthwhile investigating the case $|k_n| = 1$ in (1.1), and the author is sure that, by means of his new technique, many new combinatorial identities can be obtained, and his Ph.D. students are already working on this subject. This technique, as exposed in the *a/m* abstract, proceed as follows: let

$$(1.2) \qquad \begin{cases} P(x) = x^n + k_1 x^{n-1} + \cdots + k_{n-1}x + k_n, \quad k_i \in Z \quad (i = 1, \cdots, n) \\ P(w) = 0, \quad w \in R. \end{cases}$$

Let $Q(w)$ be the algebraic number field over Q, obtained by adjunction of w to Q. Let further

$$(1.3) \qquad \begin{cases} e = a_1 + a_2 w + \cdots + a_n w^{n-1}, \\ a_i \in Q, \quad (i = 1, \cdots, n) \end{cases}$$

where e is explicitly stated. By means of Euler's generating functions, one calculates first the positive powers of e, explicitly, viz.

$$(1.4) \qquad \begin{cases} e^m = b_{1,m} + b_{2,m}w + \cdots + b_{n,m}w^{n-1}, \\ b_{i,m} \in Q, \quad (i = 1, \cdots, n; m = 0, 1, \cdots) \end{cases}$$

$$(1.5) \qquad \begin{cases} e^{-m} = c_{1,m} + c_{2,m}w + \cdots + c_{n,m}w^{n-1}, \\ c_{i,m} \in Q, \quad (i = 1, \cdots, n; \ m = +1, 2, \cdots). \end{cases}$$

Then from

$$(1,6) \qquad \begin{cases} 1 = e^m \cdot e^{-m} = g_{1,m} + g_{2,m}w + \cdots + g_{n,m}w^{n-1} \\ g_{i,m} \in Q, \quad (i = 1, \cdots, n; m = 0, 1, \cdots) \end{cases}$$

one obtains, by comparison of coefficients of powers of w, the necessary identities, which usually involve n's order determinant with combinatorial coefficients (or their linear combinations) as entries. Thus, in [1,d)] the author obtained the combinatorial identity

$$(1.7) \qquad \begin{cases} \left(\sum_{k=0} \binom{m-k-1}{2k-1+2s} \right) = \left[\sum_{i=0} (-1)^i \binom{n-3-2i}{i} \right]^2 \\ \qquad - \sum_{i=0} (-1)^i \binom{n-4-2i}{i} \sum_{i=0} (-1)^i \binom{n-2-2i}{i} \ ; \\ m = \left[\frac{n}{2} \right]; \qquad 2s = n - 2m; \quad n = 4, 5, \cdots . \end{cases}$$

2. THE GENERATING POLYNOMIAL

A polynomial over Z of the form

$$(2.1) \qquad P_n(x) = (x - D_1)(x - D_2) \cdots (x - D_n) - d, \quad n \geq 2,$$

has been investigated by the author [1,g)] for the purpose of constructing periodic Jacobi-Perron algorithm, and by the author and Hasse [1,h)] for the purpose of obtaining $n - 1$ independent units in a functional algebraic number field of degree n. In this paper, in order to obtain the natural generalization of two-dimensional Fibonacci numbers and a new formula for the original ones, we shall investigate the case $n = 2$ of (2.1). The case $n = 3$ has been investigated by my Ph.D. student Seeder [5], for the purpose of obtaining combinatorial identities. We are now taking the liberty of marking the following

Statement. The generating polynomial for the natural generalization of the original Fibonacci numbers to the dimension two, has the form

$$(2.2)\qquad \begin{cases} F_2(x) = (x-D_1)(x-D_2)-d; \\ D_1, D_2 \in \mathbb{Z}; \quad d \in N. \\ D_1 > D_2; \quad D_1 - D_2 \equiv 0 \pmod d; \\ D_1 - D_2 \equiv 1(2); \quad d \neq m^2; \quad m \in \mathbb{Z} \end{cases}$$

The last two restrictions on $F_2(x)$ from (2.2) are chosen for convenience sake, as the reader will see later, and can, generally, be dropped for the definition of $F_2(x)$. From $D_1 - D_2 \equiv 1(2)$ would follow, since $d \mid D_1 - D_2$, that d is odd. The restriction $d \neq m^2$ is convenient in another context; both are not necessary conditions. Since $F_2(x) = x^2 - (D_1 + D_2)x + D_1D_2 - d$, the two roots of $F_2(x)$ are

$$(2.3)\qquad \begin{cases} w_1 = \dfrac{D_1 + D_2 + \sqrt{(D_1-D_2)^2+4d}}{2}; \quad w_2 = \dfrac{D_1 + D_2 - \sqrt{(D_1-D_2)^2+4d}}{2} \\ w_1, w_2 \in \mathbb{R}; \quad w_1 > w_2 . \end{cases}$$

We now prove

Lemma 1. $(D_1 - D_2)^2 + 4d$ is not a perfect square.

Proof. The lemma holds, as we shall see, even without the restrictions $D_1 - D_2 \equiv 1(2)$, $d \neq m^2$. The other restrictions of (2.2) must remain valid. We have

$$(2.4)\qquad D_1 - D_2 = td, \quad t \in \mathbb{Z} - \{0\}.$$

For $|t| = 1$, we have $(D_1 - D_2)^2 = d^2$, $(D_1 - D_2)^2 + 4d = d^2 + 4d$, and $(d+1)^2 < d^2 + 4d < (d+2)^2$.

For $|t| > 1$, we have $(D_1 - D_2)^2 = t^2d^2$,

$$t^2d^2 < t^2d^2 + 4d < t^2d^2 + 2|t|d + 1 = (|t|d+1)^2,$$

$$(|t|d)^2 < (D_1 - D_2)^2 + 4d < (|t|d+1)^2.$$

This proves the Lemma 1 completely. From Lemma 1, we immediately derive

Theorem 1. The polynomial

$$F_2(x) = (x-D_1)(x-D_2)-d; \quad D_1 - D_2 \in \mathbb{Z}; \quad d \in N; \quad D_1 > D_2; \quad D_1 - D_2 \equiv 0 \pmod d$$

is irreducible over Q (over $\mathbb{Z}$). The roots of $F_2(x)$ are real quadratic irrationals.

Notation 1. The greater of the two roots of $F_2(x)$ will be denoted by

$$(2.5)\qquad w = w_1 = \frac{D_1 + D_2 + \sqrt{(D_1-D_2)^2+4d}}{2}.$$

For later purposes we shall need the expansion of w as a simple continued periodic fraction. We have from (2.2), (2.4), (2.5)

$$(2.6)\qquad (w-D_1)(w-D_2) = d,$$

and make the restriction

$$(2.7)\qquad d \neq 1.$$

Then, as the reader can easily verify,

$$w = D_1 + \frac{1}{x_1}, \quad x_1 = \frac{1}{w-D_1} = \frac{w-D_2}{(w-D_1)(w-D_2)} = \frac{w-D_2}{d} = \frac{D_1-D_2}{d} + \frac{w-D_1}{d} = \frac{D_1-D_2}{d} + \frac{1}{x_2};$$

$$x_2 = \frac{d}{w-D_1} = \frac{d(w-D_2)}{(w-D_1)(w-D_2)} = w - D_2 = D_1 - D_2 + w - D_1 = D_1 - D_2 + \frac{1}{x_3} = D_1 - D_2 + \frac{1}{x_1}.$$

We have thus obtained: $x_1 = x_3$

$$(2.8)\qquad w = \left[D_1, \overline{\frac{D_1-D_2}{d}, D_1 - D_2}\right]$$

If we now drop restriction (2.7), we obtain

(2.9) $$w = [D_1, \overline{D_1 - D_2}]\,; \quad d = 1.$$

If we set

(2.10) $$\left\{\begin{array}{l} D_1 = 1,\ D_2 = 0;\ d = 1, \text{ we obtain} \\ x^2 - x - 1 = 0;\quad w^2 - w - 1 = 0; \\ w = \dfrac{1+\sqrt{5}}{2} = [\overline{1}]\,. \end{array}\right.$$

Thus formula (2.9) is valid also for the conditions of (2.10). Formula (2.10) leads, as was mentioned, to the original Fibonacci numbers. We return to the original case. As is known, a unit in

$$Q(w) = Q(\sqrt{(D_1 - D_2)^2 + 4d}$$

is given by

$$e_1 = x_1x_2 = \frac{d}{(w - D_1)^2}\quad ,$$

and since, from (2.6)

$$\frac{(w - D_1)^2(w - D_2)^2}{d^2} = 1,$$

we obtain

(2.11) $$e_1 \cdot 1 = e = \frac{(w - D_2)^2}{d} \text{ is a unit in } Q(w).$$

If $d = 1$, $w - D_2 \in Q(w)$, so that

(2.11a) $$w - D_2 \text{ is a unit in } Q(w),\ d = 1.$$

We shall, for the time being, eliminate the case $d = 1$, but shall return to it later. That

$$e = \frac{(w - D_2)^2}{d}\,,\quad e > 1$$

is a unit in $Q(w)$ can also be proved directly; we have

(2.12) $$\left\{\begin{array}{l} e = \dfrac{w^2 - 2D_2w + D_2^2}{d} = \dfrac{(D_1 + D_2)w - D_1D_2 + d - 2D_2w + D_2^2}{d} \\ \quad = \dfrac{-D_2(D_1 - D_2) + (D_1 - D_2)w + d}{d}\,; \end{array}\right.$$

thus e is an integer, since $D_1 - D_2 \equiv 0(d)$.

We further have

$$\begin{aligned} N(e) &= \frac{(N(w - D_2))^2}{d^2} = \frac{((w_1 - D_2)(w_2 - D_2))^2}{d^2} \\ &= d^{-2}\left(\frac{(D_1 - D_2) + \sqrt{(D_1 - D_2)^2 + 4d}}{2}\cdot\frac{(D_1 - D_2) - \sqrt{(D_1 - D_2)^2 + 4d}}{2}\right)^2 \\ &= d^{-2}\cdot d^2 = 1. \end{aligned}$$

We shall operate, in the sequel, only with the unit

$$e = \frac{(w - D_2)^2}{d}$$

regardless of whether e is fundamental or not, though this question could be easily answered. Since e is in the ring $R[w]$, we also do not need to construct a basis for $Q(w)$, and shall operate with integers of the form

(2.13) $$\beta = x + yw;\quad x,y \in \mathbb{Z}.$$

A last question remains to be resolved, viz.: are there indeed infinitely many real quadratic fields of the form $Q(\sqrt{(D_1 - D_2)^2 + 4d})$? To prove this, let us presume

$$(2.14)\qquad \begin{cases} D_1 - D_2 \equiv 1(d); & d \neq m^2, \quad m \in \mathbb{Z}, \\ D_1 - D_2 = td; & t \in \mathbb{Z};\ t \text{ fixed.} \end{cases}$$

Then

$$(D_1 - D_2)^2 + 4d = t^2 d^2 + 4d.$$

Now, Erdös [4] has proved, that for infinitely many n,

$$t^2 n^2 + 4n, \quad (tn \equiv 1(2),\ n \neq m^2,\ t, n, m \in \mathbb{Z})$$

has no square factor. This proves that there are infinitely many real quadratic fields of the form

$$Q(\sqrt{(D_1 - D_2)^2 + 4d}).$$

3. THE POWERS OF e

In this chapter we shall give formulas for the explicit calculation of e^n and e^{-n}, $(n = 0, 1, \cdots)$. This is the central result of this paper from which the new formula for the original Fibonacci numbers will be derived.

We have from (2.12)

$$e = \frac{-D_2(D_1 - D_2) + d + (D_1 - D_2)w}{d},$$

and with $D_1 - D_2 = td$, we obtain

$$(3.1)\qquad e = -D_2 t + 1 + tw.$$

From $w^2 = (D_1 + D_2)w - D_1 D_2 + d$, we obtain, with $D_1 = D_2 + td$,

$$(3.2)\qquad w^2 = -(D_2^2 + D_2 dt) + d + (2D_2 + dt)w.$$

One calculates easily from (3.1), taking into account (3.2)

$$(3.3)\qquad e^2 = -D_2 t(dt^2 + 2) + dt^2 + 1 + (dt^3 + 2t)w.$$

We now denote

$$(3.4)\qquad e^n = x_n + y_n w, \quad n = 0, 1, \cdots.$$

With (3.1), (3.3) we have

$$(3.5)\qquad \begin{cases} x_0 = 1, \ y_0 = 0; \quad x_1 = -D_2 t + 1, \ y_1 = t; \\ x_2 = -D_2(dt^3 + 2t) + dt^2 + 1; \ y_2 = dt^3 + 2t. \end{cases}$$

From (3.4), (3.1), we further obtain

$$e^{n+1} = e^n \cdot e = (x_n + y_n w)[(-D_2 t + 1) + tw].$$

An easy calculation, taking into account (3.2), yields

$$\begin{aligned} e^{n+1} = x_{n+1} + y_{n+1} w = (-D_2 t+1)x_n + (-D_2^2 t - D_2 dt^2 + dt)y_n \\ + [(-D_2 t+1)y_n + (2D_2 t + dt^2)y_n + tx_n]\, w, \end{aligned}$$

hence

$$(3.6)\qquad \begin{cases} x_{n+1} = (-D_2 t + 1)x_n + (-D_2^2 t - D_2 dt^2 + dt)y_n, \\ y_{n+1} = tx_n + (D_2 t + dt^2 + 1)y_n. \end{cases}$$

From the second formula of (3.6) we obtain

$$y_{n+2} = tx_{n+1} + (D_2 t + dt^2 + 1)y_{n+1},$$

and substituting here the value of x_{n+1} from (3.6),

$$y_{n+2} = (-D_2 t+1)tx_n + (-D_2^2 t^2 - D_2 dt^3 + dt^2)y_n + (D_2 t + dt^2 + 1)y_{n+1}.$$

Substituting here the value of tx_n from the second formula of (3.6), we obtain

$$y_{n+2} = (-D_2 t+1)[y_{n+1} - (D_2 t + dt^2 + 1)y_n] + (-D_2^2 t^2 - D_2 dt^3 + dt^2)y_n + (D_2 t + dt^2 + 1)y_{n+1},$$

and, after simple calculations

(3.7)
$$y_{n+2} = (dt^2+2)\,y_{n+1} - y_n\,.$$

From (3.6) we obtain

$$x_n = \frac{1}{t}\,[y_{n+1} - (D_2t + dt^2 + 1)y_n]\,,$$

or, raising the index by one,

$$x_{n+1} = \frac{1}{t}\,[y_{n+2} - (D_2t + dt^2 + 1)y_{n+1}]\,,$$

and substituting here the value of y_{n+2} from (3.7),

$$x_{n+1} = \frac{1}{t}\,[(dt^2+2)y_{n+1} - y_n - (D_2t + dt^2 + 1)y_{n+1}]$$

$$x_{n+1} = \frac{1}{t}\,[(-D_2t+1)y_{n+1} - y_n]$$

$$x_{n+1} = -D_2y_{n+1} + t^{-1}(y_{n+1} - y_n),$$

(3.8)
$$x_n = D_2y_n + t^{-1}(y_n - y_{n-1}).$$

Thus

(3.9)
$$e^n = [-D_2y_n + t^{-1}(y_n - y_{n-1})] + y_n w\,,$$

and, to complete the calculation of e we have to calculate y_n. This is done by means of the recurrency formula (3.7). We obtain, taking into account the values of y_0 and y_1 from (3.5)

$$\sum_{n=0}^{\infty} y_nu^n = y_0 + y_1u + \sum_{n=2}^{\infty} y_nu^n = tu + \sum_{n=0}^{\infty} y_{n+2}u^{n+2} = tu + \sum_{n=0}^{\infty} [(dt^2+2)y_{n+1} - y_n]u^{n+2}$$

$$= tu - u^2\sum_{n=0}^{\infty} y_nu^n + (dt^2+2)u\sum_{n=0}^{\infty} y_{n+1}u^{n+1}$$

$$= tu - u^2\sum_{n=0}^{\infty} y_nu^n + (dt^2+2)u\left[\left(\sum_{n=0}^{\infty} y_nu^n\right) - y_0u^0\right]$$

$$= tu - u^2\sum_{n=0}^{\infty} y_nu^n + (dt^2+2)u\sum_{n=0}^{\infty} y_nu^n\,.$$

We have obtained,

$$\sum_{n=0}^{\infty} y_nu^n = tu - u^2\sum_{n=0}^{\infty} + (dt^2+2)u\sum_{n=0}^{\infty} y_nu^n$$

(3.10)
$$\sum_{n=0}^{\infty} y_nu^n = \frac{tu}{1 - au + u^2}\,, \qquad a = dt^2 + 2.$$

From (3.10) we obtain, for u sufficiently small

$$\sum_{n=0}^{\infty} y_nu^n = tu\sum_{k=0}^{\infty} (au - u^2)^k$$

(3.11)
$$\sum_{n=0}^{\infty} y_nu^n = t\sum_{k=0}^{\infty} u^{k+1}(a-u)^k\,.$$

Collecting on the right side of (3.11) powers of u^n, we obtain, by comparison of coefficients, taking $k = n-1, n-2, \cdots,$

$$y_n = t\left[a^{n-1} - \binom{n-2}{1}a^{n-3} + \binom{n-3}{2}a^{n-5} - + \cdots\right];$$

$$y_n = t\sum_{i=0}(-1)^i\binom{n-1-i}{i}a^{n-1-2i},$$

and finally

(3.12) $$y_n = t\sum_{i=0}(-1)^i\binom{n-1-i}{i}(dt^2+2)^{n-1-2i}, \quad y_0 = 0;\; y_1 = t;\; n = 2, 3, \cdots.$$

From (3.8) and (3.12) we now also obtain the value of x_n, viz.

(3.13) $$\begin{cases} x_n = -D_2t\sum_{i=0}(-1)^i\binom{n-1-i}{i}(dt^2+2)^{n-1-2i} \\ \quad + \sum_{i=0}(-1)^i\left[\binom{n-1-i}{i}(dt^2+2)^{n-1-2i} - \binom{n-2-i}{i}(dt^2+2)^{n-2-2i}\right], \\ x_0 = 1;\quad x_1 = -D_2t+1;\quad x_2 = -D_2(dt^3+2t)+dt^2+1;\quad n = 3, 4, \cdots. \end{cases}$$

We shall now proceed to calculate the negative powers of e and use a Kunstgriff for this purpose. We remember that

$$w = w_1;\quad w_1 + w_2 = D_1 + D_2.$$

We further have

$$e^{-n} = \frac{1}{x_n + y_n w} = \frac{x_n + y_n w_2}{(x_n + y_n w_1)(x_n + y_n w_2)}.$$

Now, the whole trick consists of

$$N(e^n) = N(x_n + y_n w) = (x_n + y_n w_1)(x_n + y_n w_2);$$

but $N(e^n) = (N(e))^n = 1^n = 1$, so that

$$e^{-n} = x_n + y_n w_2 = x_n + y_n(D_1 + D_2 - w)$$

$$e^{-n} = x_n + y_n(D_1 + D_2) - y_n w.$$

But from (3.8), $x_n = -D_2y_n + t^{-1}(y_n - y_{n-1})$, so that finally

(3.14) $e^{-n} = D_1y_n + t^{-1}(y_n - y_{n-1}) - y_n w;\quad y_0 = 0;\; y_1 = t;\; n = 2, 3, \cdots;\; y_n$ from (3.12).

The reader will easily verify that the norm equation of e^n yields

(3.14a) $$x_n^2 + (D_1 + D_2)x_n y_n + (D_1D_2 - d)y_n^2 = 1.$$

4. THE "NEW" FORMULA

We return to the generating polynomial of the original Fibonacci numbers, $P(x) = x^2 - x - 1$. We have

(4.1) $P(w_1) = P(w_2) = 0;\; w_1 = \dfrac{1+\sqrt{5}}{2},\; w_2 = \dfrac{1-\sqrt{5}}{2},\; w^2 - w - 1 = 0;\; w^2 = w + 1;\; w = w_1$

In $Q(w)$, w is a (fundamental) unit; we shall calculate its non-negative integral powers.

(4.2) $$w^n = g_n + f_n w;\quad g_0 = 1;\quad g_1 = 0;\quad f_0 = 0;\quad f_1 = 1.$$

Multiplying in (4.2) both sides of w, we obtain

$$w^{n+1} = g_n w + f_n w^2 = g_n w + f_n(w+1),\quad w^{n+1} = f_n + (g_n + f_n)w = g_{n+1} + f_{n+1}w$$

$$g_{n+1} = f_n;\quad f_{n+1} = f_n + g_n = f_n + f_{n-1}$$

(4.3) $$w^n = f_{n-1} + f_n w$$

(4.4) $$f_{n+2} = f_n + f_{n+1}.$$

Since $w^2 = g_2 + f_2 w = 1 + w$, we have $f_2 = 1$, so that (4.2), (4.4) and $f_2 = 1$ yield,

$$f_{n+2} = f_n + f_{n+1}; \quad f_1 = f_2 = 1; \quad n = 1, 2, \cdots$$

which shows that the f_n are the original Fibonacci numbers,

(4.5) $$f_n = F_n; \quad n = 1, 2, \cdots.$$

If we set in (2.2)

$$D_1 = 1; \quad D_2 = 0; \quad d = 1; \quad t = 1,$$

we obtain, from (2.11), (3.4), (4.2), (4.5) and (3.12)

$$e^n = w^{2n} = x_n + y_n w = g_{2n} + f_{2n} w, \quad F_{2n} = y_n;$$

(4.6) $$\underline{\underline{F_{2n} = \sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-2i},}}$$

since $dt^2 + 2 = 3$.

(4.6) is the new, and surprising, beautiful formula for F_{2n}; F_{2n+1} is then obtained from the relation

$$F_{2n+1} = F_{2n+2} - F_{2n} = \left(\sum_{i=0}^{\infty} (-1)^i 3^{n-2i} \binom{n-1}{i} \right) - \sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-2i},$$

so that, by the new approach to Fibonacci numbers, we obtain the sequence (which is, of course, identical with the original one)

(4.7) $$\begin{cases} F_1 = F_2 = 1; \quad F_{2n} = \sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-2i}; \quad n = 2, 3, \cdots; \\ F_{2n+1} = \left(\sum_{i=0} (-1)^i \binom{n-i}{i} 3^{n-2i} \right) - \left(\sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-i} \right); n = 1, 2, \cdots \end{cases}$$

In (4.7), for $n = 1$ in F_{2n+1}, we have to define $\binom{0}{0} \stackrel{\text{def}}{=} 1$.

From (4.3), we have, with (4.5)

$$w^n = F_{n-1} + F_n w.$$

Now. since $w^2 - w - 1 = 0$, we have

$$N(w) = -1,$$

so that

$$\begin{aligned} N(w^n) = (-1)^n = N(F_{n-1} + F_n w) &= (F_{n-1} + F_n w_1)(F_{n-1} + F_n w_2) \\ &= F_{n-1}^2 + (w_1 + w_2) F_{n-1} F_n + F_n^2 w_1 w_2 = F_{n-1}^2 + F_{n-1} F_n - F_n^2 \\ &= F_{n-1}^2 + F_{n-1}(F_{n+1} - F_{n-1}) - F_n^2 = F_{n-1} F_{n+1} - F_n^2, \end{aligned}$$

$$F_n^2 - F_{n-1} F_{n+1} = (-1)^{n+1},$$

a well known formula.

The analogue for the generalized generating polynomial $F_2(x)$ from (2.2) is obtained from (3.4), with $N(e) = 1$, viz.

$$x_n^2 + (D_1 + D_2) x_n y_n + (D_1 D_2 - d) y_n^2 = 1,$$

which solves the Diophantine equation

$$x^2 + (D_1 + D_2) xy + (D_1 D_2 - d) y^2 = 1,$$

$$D_1 > D_2; \quad D_1 - D_2 \equiv 0(d); \quad d, D_1, D_2 \in Z; \quad d \geqslant 1.$$

5. ALTNEULAND*—AN EPILOGUE

The new Formula for the original Fibonacci numbers, as the author has called it with unforgivable self-styled praise, is actually an old formula which could be achieved by elementary means, as was kindly remarked to the author in a private correspondence by Professor Verner E. Hoggatt, Jr., of San Jose State University. Here is the way it can be obtained from the original Fibonacci numbers:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + F_{2n-1} = 2F_{2n} + F_{2n} - F_{2n-2};$$

(5.1) $$F_{2n+2} = 3F_{2n} - F_{2n-2}\,.$$

Eq. (5.1) leads to the generating polynomial

(5.2) $$x^2 - 3x + 1 = 0,$$

and from (5.2) the new formula for F_{2n+2} is easily obtained by the use of Euler's generating functions, as used in this paper. But finding a new formula for F_{2n+2} was not the idea of this paper, as was pointed out in the introduction. The aim was two-fold—first finding the most natural generalization for Fibonacci numbers, of which the original ones would be a special case; second—to demonstrate the powerful use of units to finding combinatorial identities, since, after all, what we have found—and again, this may be considered Altneuland—is the combinatorial identity

(5.3) $$\sum_{i=0} \binom{2n-1-i}{i} = \sum_{i=0} (-1)^i \binom{n-1-i}{i} 3^{n-1-2i}\,.$$

Besides the technique used in this paper, the author has found a new, and, as he believes, powerful different technique by using units in algebraic functional fields of any degree for finding new combinatorial identities of higher dimension which surely cannot be proved by elementary combinatorial means. These new results will appear in a book by the author which is now in preparation.

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*Title of the famous novel by the Austrian writer, Theodor Herzl.

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LETTER TO THE EDITOR

Dear Editor:

I am teaching a survey course at the Pennsylvania State University. After two days of studying the elementary properties of the Fibonacci sequence, I asked my class to write a poem about Fibonacci. One very talented student submitted the following:

FIBONACCI'S PARTY
by Cynthia Ellis

The great mathematician Fibonacci
Went out to the market and bought a Hibachi.
He decided to give a small Bar-B-Que
For himself, his wife, and a good friend or two.

So he called his friend Joe and he asked him to come
With a small jug of wine or a bottle of rum.
"My wife (one) and I (one) make two" figured he,
"And with Joseph attending, the total is three."

But then the telephone rang in the hall:
His parents would be there, making five guests in all.
And his wife told him also her parents were coming.
With sister Loretta—now eight was his summing.

But, oh, he'd forgotten Joe's girlfriend Eileen.
With her and her family the total's thirteen.
And Loretta brings friends to wherever there's fun.
So he counted it up and he got twenty-one.

Just then he remembered the neighbors next door.
They'd certainly be there to make thirty-four.
And then his club's football teams pulled in the drive.
And he tore at his hair as he thought "Fifty-five!"

While out in the street he saw line after line
Of neighborhood moochers to make eighty-nine.
And 'round from the alley there came at a trot
His boss and co-workers, the whole bloomin' lot.

Fib went to the gameroom and sat on the floor
And figured the total as one-forty-four.
So he crawled to the bar and swalled a dose
And started to wonder how three grew to gross.

So he pulled out his list and he started to count,
Carefully writing down every amount.
And discovered the sequence that now bears his name,
Thanks to the party where everyone came.

I hope you like the poem and decide to publish it.

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