A FIBONACCI PROPERTY OF WYTHOFF PAIRS

ROBERT SILBER
North Carolina State University, Raleigh, North Carolina 27607

In this paper we point out another of those fascinating "coincidences" which are so characteristically associated with the Fibonacci numbers. It occurs in relation to the so-called safe pairs (a_n, b_n) for Wythoff's Nim [1, 2, 3]. These pairs have been extensively analyzed by Carlitz, Scoville and Hoggatt in their researches on Fibonacci representations [4, 5, 6, 7], a context unrelated to the game of nim. The latter have carefully established the basic properties of the a_n and b_n, so that even though that which we are about to report is not described in their investigations, it is a ready consequence of them. For convenience and for reasons of precedence, we refer to the pairs (a_n, b_n) as *Wythoff pairs.*

The first forty Wythoff pairs are listed in Table 1 for reference. We recall that the pairs are defined inductively as follows: $(a_1, b_1) = (1, 2)$, and, having defined $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$, a_{n+1} is defined as the smallest positive integer not among $a_1, b_1, a_2, b_2, \cdots, a_n, b_n$ and then b_{n+1} is defined as $a_{n+1} + (n + 1)$. Each positive integer occurs exactly once as a member of some Wythoff pair, and the sequences $\{a_n\}$ and $\{b_n\}$ are (strictly) increasing.

Wythoff [1] showed that $a_n = [n\alpha]$ and $b_n = [n\alpha^2]$, α being the golden ratio $(1 + \sqrt{5})/2$; a more elegant proof of this appears in [3]. This connection of the Wythoff pairs with the golden ratio suggests to any "Fibonaccist" that the Fibonacci numbers are not very far out of the picture. The work of Carlitz *et. al.* that we have mentioned shows that the a_n and b_n play a fundamental part in the analysis of the Fibonacci number system. We now look at another connection with Fibonacci numbers.

Table 1
The First Forty Wythoff Pairs

n	a_n	b_n	n	a_n	b_n	n	a_n	b_n	n	a_n	b_n
1	1	2	11	17	28	21	33	54	31	50	81
2	3	5	12	19	31	22	35	57	32	51	83
3	4	7	13	21	34	23	37	60	33	53	86
4	6	10	14	22	36	24	38	62	34	55	89
5	8	13	15	24	39	25	40	65	35	56	91
6	9	15	16	25	41	26	42	68	36	58	94
7	11	18	17	27	44	27	43	70	37	59	96
8	12	20	18	29	47	28	45	73	38	61	99
9	14	23	19	30	49	29	46	75	39	63	102
10	16	26	20	32	52	30	48	78	40	64	104

In examining Table 1, it is interesting to observe that the first few Fibonacci numbers occur paired with other Fibonacci numbers:

$(a_1, b_1) = (1,2)$, $(a_2, b_2) = (3,5)$, $(a_5, b_5) = (8, 13)$, $(a_{13}, b_{13}) = (21, 34)$, $(a_{34}, b_{34}) = (55, 89)$.

It is not difficult to establish that this pattern continues throughout the sequence of Wythoff pairs, using the fact that $\lim (b_n/a_n) = \alpha$ and also $\lim (F_{n+1}/F_n) = \alpha$. However, an almost immediate proof can be had, based on Eq. (3.5) of [4], which states that

$$a_n + b_n = a_{b_n} \qquad (1)$$

for each positive integer n. We defer the proof momentarily, it being our intention to state and prove a generalization of the foregoing.

Clearly there are many Wythoff pairs whose members are not Fibonacci numbers; the first such is $(a_3, b_3) = (4,7)$. The pair (4,7) can be used to generate a Fibonacci sequence in the same way that $(a_1, b_1) = (1,2)$ can be considered to determine the usual Fibonacci numbers; we take $G_1 = 4$, $G_2 = 7$, $G_{n+2} = G_{n+1} + G_n$. The first few terms of the resulting Fibonacci sequence are

$$4, 7, 11, 18, 29, 47, \cdots.$$

It is (perhaps) a bit startling to observe that

$$(a_3, b_3) = (4, 7), \qquad (a_7, b_7) = (11, 18), \qquad (a_{18}, b_{18}) = (29, 47).$$

The pair $(a_4, b_4) = (6, 10)$ similarly generates a Fibonacci sequence

$$6, 10, 16, 26, 42, 68, \cdots$$

and sure enough

$$(a_4, b_4) = (6, 10), \qquad (a_{10}, b_{10}) = (16, 26), \qquad (a_{26}, b_{26}) = (42, 68).$$

It is time for our first theorem.

Theorem. Let $G_1, G_2, G_3, \cdots$ be the Fibonacci sequence generated by a Wythoff pair (a_n, b_n). Then every pair $(G_1, G_2), (G_3, G_4), (G_5, G_6), \cdots$ is again a Wythoff pair.

Proof. By construction, every Wythoff pair satisfies

$$a_k + k = b_k . \tag{2}$$

Consider the first four terms of the generated Fibonacci sequence:

$$a_n, b_n, a_n + b_n, a_n + 2b_n .$$

According to Eq. (1)

$$a_n + b_n = a_{b_n},$$

so that

$$a_n + 2b_n = a_{b_n} + b_n .$$

Equation (2) with $k = b_n$ gives

$$a_{b_n} + b_n = b_{b_n} ,$$

so that the four terms under consideration are in fact

$$a_n, b_n, a_{b_n}, b_{b_n} .$$

Thus we have proven that in general (G_3, G_4) is a Wythoff pair when (G_1, G_2) is. But (G_5, G_6) can be considered as consisting of the third and fourth terms of the Fibonacci sequence generated by (G_3, G_4), and the latter is already known to be a Wythoff pair. In this way, the theorem follows by induction.

Thus we see that each Wythoff pair generates a sequence of Wythoff pairs; the pairs following the first pair of the sequence will be said to be generated by the first pair. We define a Wythoff pair to be *primitive* if no other Wythoff pair generates it. It is clear that if (a_m, b_m) generates (a_n, b_n), then $m < n$. For this reason, one can determine the first few primitive pairs by the following algorithm, analogous to Eratosthenes' sieve. The first pair (1,2) is clearly primitive. All those generated by (1,2) are eliminated (up to some specified point in the table). The first pair remaining must again be primitive, and all pairs generated by that primitive are eliminated. The process is repeated.

The first few primitive pairs so determined are pair numbers

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, \cdots$$

which we recognize at once to be the sequence

$$a_1, a_2, a_3, \cdots.$$

This occasions our next theorem.

Theorem. A Wythoff pair (a_n, b_n) is primitive if and only if $n = a_k$ for some positive integer k.

Proof. We have seen that the terms of the Fibonacci sequence generated by any Wythoff pair (a_n, b_n) are of the form

$$a_n, b_n, a_{b_n}, b_{b_n}, a_{b_{b_n}}, b_{b_{b_n}}, \cdots .$$

From this it is obvious that any non-primitive pair (a_n, b_n) must have $n = b_k$ for some positive integer k, which makes every pair (a_n, b_n) with $n = a_k$ a primitive pair.

On the other hand, the sequence

$$a_n, b_n, a_{b_n}, b_{b_n}, \cdots$$

generated by (a_n, b_n) shows clearly that each pair (a_{b_k}, b_{b_k}) is generated by (a_k, b_k); thus every primitive pair (a_n, b_n) must have $n = a_k$.

This theorem shows that the number of primitive pairs is infinite, and has the following corollary.

Corollary. There exists a sequence of Fibonacci sequences which simply covers the set of positive integers.

An interesting property of the primitive pairs turns up when we calculate successively the determinants

$$\begin{vmatrix} a_n & b_n \\ a_1 & b_1 \end{vmatrix},$$

restricting our attention to those (a_n, b_n) which are primitive. We find that

$$\begin{vmatrix} 4 & 7 \\ 1 & 2 \end{vmatrix} = 1, \qquad \begin{vmatrix} 6 & 10 \\ 1 & 2 \end{vmatrix} = 2,$$

$$\begin{vmatrix} 9 & 15 \\ 1 & 2 \end{vmatrix} = 3, \qquad \begin{vmatrix} 12 & 20 \\ 1 & 2 \end{vmatrix} = 4,$$

and so on. This suggests that the value of the determinant applied to the k^{th} primitive is $k - 1$. By the foregoing theorem, we know that the k^{th} primitive is in fact (a_{a_k}, b_{a_k}), so the suggested identity becomes

$$2a_{a_k} - b_{a_k} = k - 1,$$

which follows readily from Eq. (3.2) of [4].

We conclude by interpreting our results in terms of the findings in [4] and [6]. According to the latter, the Wythoff pairs (a_n, b_n) are those pairs of positive integers with the following two properties: first, the canonical Fibonacci representation of b_n is exactly the left shift of the canonical Fibonacci representation of a_n, and second, the right-most 1 appearing in the representation of a_n occurs in an even numbered position. (In base 2 this would be analogous to saying that $b_n = 2a_n$ and that the largest power of 2 which divides a_n is odd). No two 1's appear in succession in the representations of a_n and b_n. If we add a_n and b_n, each 1 in the representation of b_n will combine with its shift in the representation of a_n to yield a 1 in the position immediately to the left of the added pair, since $F_n + F_{n-1} = F_{n+1}$. This means that $b_n + a_n$ has a representation which is exactly the left shift of b_n. By exactly the same reasoning, $(b_n + a_n) + b_n$ has a representation which is exactly the left shift of $b_n + a_n$, and so forth. *Hence, the Fibonacci sequence generated by any Wythoff pair, when expressed canonically in the Fibonacci number system, consists of consecutive left shifts of the first term of the sequence.* In the simplest case, the pair (1,2) generates the usual Fibonacci sequence, which in the Fibonacci number system would be expressed

10, 100, 1000, 10000, 100000, ⋯

and the generated pairs would be

(10, 100), (1000, 10000), ⋯

which have the requisite properties that each b_n is the left shift of a_n and that each a_n has its right-most 1 in an even-numbered position. The next case corresponds to the sequence generated by (4, 7); 4, 7, 11, 18, ⋯. In Fibonacci, this appears as

$$1010,\ 10100,\ 101000,\ 1010000,\ \cdots.$$

This procedure can be traced back an additional step to the index n of the pair (a_n, b_n). Doing so provides in addition a simple interpretation of the primitive pairs in terms of Fibonacci representations. There is a prescription in [6] for generating a Wythoff pair (a_n, b_n) from its index n, but it necessitates the so-called second canonical Fibonacci representation. For present purposes it suffices to remark that the second canonical representation of any n can be obtained by adding 1 to the usual canonical representation of $n - 1$. (For example, the canonical representation of 7 is 10100, so the second canonical representation of 8 is 10101). *The numbers* a_n *and* b_n *are then obtained from successive left shifts of the second canonical representation of* n. Thus, in the example of (4,7), we obtain the second canonical representation of 4 as 1000 + 1 = 1001 and generate

$$\begin{array}{ccccc} n & a_n & b_n & a_n + b_n = a_{b_n}, & \text{etc.} \\ 10001: & 10010 & 100100 & 1001000 & \cdots. \end{array}$$

We have seen that the primitive pairs correspond to the case $n = a_k$. It is readily established on the basis of the results in [4] that the numbers a_k are precisely those numbers whose second canonical Fibonacci representations contain a 1 in the first position (as follows: first, a number is a b_k if and only if its canonical representation contains its right-most 1 in an odd position—which is never the first—and, second, a second canonical representation fails to be canonical if and only if it contains a 1 in the first position). It follows that *the primitive pairs* (a_n, b_n) *are precisely those for which, the second canonical representation of* n *having a 1 in the first position, the canonical representation of* a_n *ends in 10 and that of* b_n *ends in 100.* Other terms of the generated Fibonacci sequence have additional zeroes, the location of any number in the sequence being exactly dependent on the number of terminal zeroes in its canonical representation.

This enables one to determine for any positive integer n exactly which primitive Wythoff pair generates the Fibonacci sequence in which that n appears, as well as the location of n in that sequence. First determine the canonical Fibonacci representation of n. The portion of the representation between the first and last 1's, inclusive, is the second canonical representation of the number k of the primitive Wythoff pair which generates the Fibonacci sequence containing n. One left shift produces a_k; another produces b_k. Counting a_k as the first term, b_k as the second, and so forth, the i^{th} term will equal n, where i is the number of zeroes prior to the first 1 in the Fibonacci representation of n. For example, let $n = 52$. In Fibonacci, 52 is represented 101010000. Since 10101 represents 8 and the representation terminates in four zeroes, 52 must be the fourth term of the Fibonacci sequence generated by the primitive pair (a_8, b_8). As confirmation we note that this particular sequence is

$$12,\ 20,\ 32,\ 52,\ 84,\ \cdots.$$

A FOOTNOTE

Because of the connection of the Wythoff pairs with Wythoff's Nim, the preceding prescription for generating Wythoff pairs is clearly also a prescription for playing Wythoff's Nim using the Fibonacci number system. This gives the Fibonacci number system a role in this game quite analogous to the role of the binary number system in Bouton's Nim [8]. The analysis of Wythoff's Nim using Fibonacci representations can be made self-contained and elementary, certainly not requiring the level of mathematical sophistication required to follow the investigations in [4, 5, 6, 7]. For the benefit of those interested in mathematical recreations, we provide this analysis in a companion paper [9]. It is interesting to note that the role of the Fibonacci number system in nim games was already anticipated by Whinihan [10] in 1963.

REFERENCES

1. W. A. Wythoff, "A Modification of the Game of Nim," *Nieuw Archief voor Wiskunde,* 2nd Series, Vol. 7, 1907, pp. 199–202.
2. A. P. Domoryad, *Mathematical Games and Pastimes,* N. Y., Macmillan Co., 1964, pp. 62–65.
3. H. S. M. Coxeter, "The Golden Section, Phyllotaxis, and Wythoff's Game," *Scripta Mathematica,* Vol. 19, 1953, pp. 135–143.
4. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 10, 1972, pp. 1–28.
5. L. Carlitz, et. al., "Lucas Representations," *The Fibonacci Quarterly,* Vol. 10, 1972, pp. 29–42.
6. L. Carlitz, et. al., "Addendum to the Paper 'Fibonacci Representations'," *The Fibonacci Quarterly,* Vol. 10, 1972, pp. 527–530.
7. L. Carlitz, R. Scoville and T. Vaughan, "Some Arithmetic Functions Related to Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 11, 1973, pp. 337–386.
8. C.L. Bouton, "Nim, A Game with a Complete Mathematical Theory," *Annals of Mathematics,* 1901–1902, pp. 35–39.
9. R. Silber, "Wythoff's Nim and Fibonacci Representations," to appear, *The Fibonacci Quarterly.*
10. M. J. Whinihan, "Fibonacci Nim," *The Fibonacci Quarterly,* Vol. 1, No. 4, 1963, pp. 9–13.

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