ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico 87131

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-340 Proposed by Phil Mana, Albuquerque, New Mexico.

Characterize a sequence whose first 28 terms are:

1779, 1784, 1790, 1802, 1813, 1819, 1824, 1830, 1841, 1847, 1852, 1858, 1869, 1875, 1880, 1886, 1897, 1909, 1915, 1920, 1926, 1937, 1943, 1948, 1954, 1965, 1971, 1976.

B-341 Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, New York.

Prove that the product $F_{2n}F_{2n+2}F_{2n+4}$ of three consecutive Fibonacci numbers with even subscripts is the product of three consecutive integers.

B-342 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Prove that $2L_{n-1}^3 + L_n^3 + 6L_{n+1}^2L_{n-1}$ is a perfect cube for $n = 1, 2, \cdots$.

B-343 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a simple expression for

$$\sum_{k=1}^{n} [F_{2k-1}F_{2(n-k)+1} - F_{2k}F_{2(n-k+1)}] .$$

B-344 Proposed by Frank Higgins, Naperville, Illinois.

Let c and d be real numbers. Find $\lim_{n \to \infty} x_n$, where x_n is defined by $x_1 = c$, $x_2 = d$, and

$$x_{n+2} = (x_{n+1} + x_n)/2 \quad \text{for} \quad n = 1, 2, 3, \cdots.$$

B-345 Proposed by Frank Higgins, Naperville, Illinois.

Let $r > s > 0$. Find $\lim_{n \to \infty} P_n$, where P_n is defined by $P_1 = r + s$ and $P_{n+1} = r + s - (rs/P_n)$ for $n = 1, 2, 3, \cdots$.

SOLUTIONS

A FIBONACCI ALPHAMETIC

B-316 Proposed by J. A. H. Hunter, Fun with Figures, *Toronto, Ontario, Canada.*

Solve the alphametic

$$\begin{array}{ccccc} & & T & W & O \\ T & H & R & E & E \\ \underline{T} & \underline{H} & \underline{R} & \underline{E} & \underline{E} \\ E & I & G & H & T \end{array}$$

Believe it or not, there must be no 8 in this!

Solution by Charles W. Trigg, San Diego, California.

$T < 5$, and no letter represents 8. There are five cases to consider.

(1) If $2T + 1 = E$, and $T = 1$, then $E = 3$, and $O = 5$.
If $H = 6$, then $W = 9$, $I = 2$, and $2 + 2R = G$, impossible.
If $H = 7$, then $W = 0$, $I = 4$, and $I + 2R = G$, impossible.
If $H = 9$, then $I = 8$, which is prohibited.

(2) If $2T + 1 = E$ and $T = 3$, then $E = 7$ and $O = 9$.
If $H = 4$, then $W = 8$, prohibited.
If $H = 5$, then $W = 9 = O$.
If $H = 6$, then $W = 0$, and $4 + 2R = G$, impossible.

(3) If $2T + 1 = E$ and $T = 4$, then $E = 9$, $O = 6$, and $H = W$.

(4) If $2T = E$ and $T = 1$, then $E = 2$ and $O = 7$.
If $H = 3$, then $W = 8$, which is prohibited.
If $H = 4$, then $W = 9$ and $I = 8$ or 9.

(5) If $2T = E$ and $T = 3$, then $E = 6$ and $O = 1$.
If $H = 4$, then $W = 1 = O$.
If $H = 2$, then $W = 9$ and $5 + 2R = G$ or $G + 10$.

Whereupon, $R = 0$, $G = 5$, and $I = 4$. Thus the unique reconstructed addition is

$$391 + 32066 + 32066 = 65423.$$

Also solved by Nancy Barta, Richard Blazej, Paul S. Bruckman, John W. Milsom, C. B. A. Peck, James F. Pope, and the Proposer.

LUCAS DIVISOR

B-317 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that L_{2n-1} is an exact divisor of $L_{4n-1} - 1$ for $n = 1, 2, \cdots$.

Solution by Gerald Bergum, Brookings, South Dakota.

Using the Binet formula together with $\alpha\beta = -1$ and $\alpha + \beta = 1$ we have

$$L_{2n}L_{2n-1} = (\alpha^{2n} + \beta^{2n})(\alpha^{2n-1} + \beta^{2n-1}) = \alpha^{4n-1} + \beta^{4n-1} + (\alpha\beta)^{2n-1}(\alpha + \beta) = L_{4n-1} - 1.$$

Also solved by M. D. Agrawal, George Berzsenyi, Richard Blazej, Wray G. Brady, Paul S. Bruckman, Ralph Garfield, Frank Higgins, Mike Hoffman, Peter A. Lindstrom, Graham Lord, Carl F. Moore, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIBONACCI SQUARE

B-318 Proposed by Herta T. Freitag, Roanoke, Virginia.

Prove that $F_{4n}^2 + 8F_{2n}(F_{2n} + F_{6n})$ is a perfect square for $n = 1, 2, \cdots$.

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

Using well known identities (see, for example, I_{21} and I_7 in Hoggatt's *Fibonacci and Lucas Numbers*) one finds that

$$F_{4n}^2 + 8F_{2n}(F_{2n} + F_{6n}) = F_{4n}^2 + 8F_{2n}(F_{4n}L_{2n}) = F_{4n}^2 + 8F_{4n}(F_{2n}L_{2n}) = F_{4n}^2 + 8F_{4n}^2$$
$$= 9F_{4n}^2 = (3F_{4n})^2.$$

Also solved by M. D. Agrawal, Gerald Bergum, Richard Blazej, Wray G. Brady, Ralph Garfield, Frank Higgins, Mike Hoffman, Peter A. Lindstrom, Graham Lord, Carl F. Moore, C. B. A. Peck, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

RERUN

B-319 Prove or disprove:

$$\frac{1}{L_2} + \frac{1}{L_6} + \frac{1}{L_{10}} + \cdots = \frac{1}{\sqrt{5}}\left(\frac{1}{F_2} - \frac{1}{F_6} + \frac{1}{F_{10}} - \cdots\right).$$

Solution (independently) by Carl F. Moore, Tacoma, Washington, and C. B. A. Peck, State College, Pennsylvania.

This problem is a restatement of the problem B-111, proposed and solved by L. Carlitz, *The Fibonacci Quarterly*, Vol. 5, No. 4 (Dec. 1967), p. 470.

Also solved by Paul S. Bruckman, Mike Hoffman, and the Proposer.

A SUM

B-320 Proposed by George Berzsenyi, Beaumont, Texas.

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m}.$$

Solution by Gerald Bergum, Brookings, South Dakota.

Using induction it is easy to show that

$$\sum_{k=0}^{2t} F_k F_{k+d} = F_{2t} F_{2t+d+1}.$$

If n is even, we have,

$$\sum_{k=0}^{n} F_k F_{k+2m} = F_n F_{n+2m+1}.$$

If n is odd, we have

$$\sum_{k=0}^{n} F_k F_{k+2m} = F_{n-1} F_{n+2m} + F_n F_{n+2m} = F_{n+1} F_{n+2m}.$$

Also solved by M. D. Agrawal, Paul S. Bruckman, Herta T. Freitag, Frank Higgins, Graham Lord, Carl F. Moore, C. B. A. Peck, James F. Pope, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

A RELATED SUM

B-321 Proposed by George Berzsenyi, Beaumont, Texas.

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m+1}.$$

Solution by Gerald Bergum, Brookings, South Dakota.

Using induction it is easy to show that

$$\sum_{k=0}^{2t} F_k F_{k+d} = F_{2t} F_{2t+d+1}.$$

If n is even, we have

$$\sum_{k=0}^{n} F_k F_{k+2m+1} = F_n F_{n+2m+2}.$$

If n is odd, we have

$$\sum_{k=0}^{n} F_k F_{k+2m+1} = F_{n-1} F_{n+2m+1} + F_n F_{n+2m+1} = F_{n+1} F_{n+2m+1}.$$

Also solved by the sovlers of B-320.

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ADVANCED PROBLEMS AND SOLUTIONS

Show that

$$L(m,n) - 25F(m,n) = 8L_{m+n} F_{m+1} F_{n+1}.$$

Solution by the Proposer.

It follows from the Binet formulas

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \qquad L_m = \alpha^m + \beta^m$$

that

$$5F_m F_n = L_{m+n} - (\alpha^m \beta^n + \alpha^n \beta^m),$$

so that

$$5F_{i+j} F_{m-i+n-j} = L_{m+n} - (\alpha^{i+j}\beta^{m-i+n-j} + \alpha^{m-i+n-j})$$
$$5F_{i+n-j} F_{m-i+j} = L_{m+n} - (\alpha^{i+n-j}\beta^{m-i+j} + \alpha^{m-i+j}\beta^{i+n-j}).$$

Hence

$$\begin{aligned} 25F_{i+j} F_{m-i+j} F_{i+n-j} F_{m-i+n-j} = L^2_{m+n} - L_{m+n}(\alpha^{i+j}\beta^{m-i+n-j} + \alpha^{m-i+n-j}\beta^{i+j} \\ + \alpha^{i+n-j}\beta^{m-i+j} + \alpha^{m-i+j}\beta^{i+n-j}) \\ + (\alpha^{2i+n}\beta^{2m-2i+n} + \alpha^{2m-2i+n}\beta^{2i+n} + \alpha^{m+2j}\beta^{m+2n-2j} + \alpha^{m+2n-2j}\beta^{m+2j}). \end{aligned}$$

It follows that

$$25F(m,n) = (m+1)(n+1)L^2_{m+n} - 4L_{m+n} F_{m+1} F_{n+1} + 2(-1)^n (n+1) F_{2m+2} + 2(-1)^m (m+1) F_{2n+2}.$$

Similarly,

$$L(m,n) = (m+1)(n+1)L^2_{m+n} + 4L_{m+n} F_{m+1} F_{n+1} + 2(-1)^n (n+1) F_{2m+2} + 2(-1)^m (m+1) F_{2n+2}.$$

Therefore,

$$L(m,n) - 25F(m,n) = 8L_{m+n} F_{m+1} F_{n+1}.$$

Also solved by P. Bruckman.

EDITORIAL REQUEST! Send in your problem proposals!

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