GOLDEN SEQUENCES OF MATRICES
WITH APPLICATIONS TO FIBONACCI ALGEBRA

JOSEPH ERCOLANO
Baruch College, CUNY, New York, New York 10010

1. INTRODUCTION

As is well known, the problem of finding a sequence of real numbers, $\{a_n\}$, $n = 0, 1, 2, \cdots$, which is both geometric $(a_{n+1} = ka_n,\ n = 0, 1, 2, \cdots)$ and "Fibonacci" $(a_{n+1} = a_n + a_{n-1},\ n = 1, 2, \cdots$, with $a_0 = 1)$ admits a solution—in fact, a unique solution. (Cf. [1]; for some extensions and geometric interpretations, see [2].) This "golden sequence" [1] is:

$$1,\ \phi,\ \phi^2, \cdots, \phi^n, \cdots,$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$, the "golden mean," and satisfies the Fibonacci equation

$$x^2 - x - 1 = 0.$$

In this paper, we pose an equivalent problem for a sequence of real, non-singular 2×2 matrices. Curiously, we will show that this problem admits an infinitude of solutions (i.e., that there exist infinitely many such "golden sequences"); that each such sequence is naturally related to each of the others (the relation given in familiar, algebraic terms of the generators of the sequences); and that these sequences are essentially the only such "golden sequences" of matrices (this, a simple consequence of a classical theorem of linear algebra). Finally, by applying two basic tools from the theory of matrices to the generators of these golden sequences, we deduce simply and naturally, some of the more familiar Fibonacci/Lucas identities [3] (including several which appear to be new); and the celebrated Binet formulas for the general terms of the Fibonacci and Lucas sequences.

2. THE DEFINING EQUATIONS

Let

$$A = \begin{pmatrix} x & y \\ u & v \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where x, y, u, v are to be determined subject to the constraint that $xv - yu \neq 0$. Clearly, a necessary and sufficient condition for the geometric sequence

$$I, A, A^2, A^3, \cdots, A^n, \cdots$$

to be "Fibonacci" is that

$$A^2 = A + I; \tag{0}$$

that is, that

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \cdot \begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} x & y \\ u & v \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

(The necessity of (0) is clear; further, (0) implies that

$$A^{n+1} = A^n + A^{n-1}, \qquad n = 1, 2, 3, \cdots,$$

so long as A is not nilpotent. This will be the case since we're restricting A to be nonsingular.) A simple calculation shows that the matrix equation in (1) is equivalent to the following system of scalar equations:

$$x^2 + yu = x + 1, \qquad xy + yv = y, \qquad xu + uv = u, \qquad yu + v^2 = v + 1, \tag{2}$$

which we write in the following more convenient form:

$$x^2 - x - 1 + yu = 0 \tag{3.1}$$

$$(x + v - 1)y = 0 \tag{3.2}$$

$$(3.3)\qquad (x+v-1)u = 0$$

$$(3.4)\qquad v^2-v-1+yu = 0.$$

We now investigate possible solution sets.

Case 1. $y = 0$. Equations (3.1), (3.4) reduce to the Fibonacci equation, implying $x = \{\phi, \phi'\}$, $v = \{\phi, \phi'\}$, where $\phi = \frac{1}{2}(1+\sqrt{5})$, $\phi' = \frac{1}{2}(1-\sqrt{5})$.

(a) If $u = 0$, solution matrices of (1) are

$$\Phi_0 = \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}, \qquad \Phi_1 = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} \phi' & 0 \\ 0 & \phi \end{pmatrix}, \qquad \Phi_3 = \begin{pmatrix} \phi' & 0 \\ 0 & \phi' \end{pmatrix}.$$

(The reader not familiar with the elementary identities involving ϕ and ϕ' is referred to either [1, 3]. The easily proved identities we will need in the sequel are

$$\phi+\phi' = 1, \quad \phi-\phi' = \sqrt{5}, \quad 2\phi-1 = \sqrt{5}, \quad \phi\cdot\phi' = -1, \quad \phi^2 = \phi+1,$$
$$\phi'^2 = \phi'+1, \quad \phi^{n+1} = \phi^n+\phi^{n-1}, \quad \phi'^{n+1} = \phi'^n+\phi'^{n-1}, \quad n = 0, \pm 1, \pm 2, \cdots.)$$

Application of the appropriate identities shows that each of the sequences

$$\{\Phi_0^n\}, \quad \{\Phi_1^n\}, \quad \{\Phi_2^n\}, \quad \{\Phi_3^n\}$$

is golden (the second and fourth of these sequences are said to be trivial).

(b) If $u \neq 0$, equation (3.3) implies $x+v = 1$, and hence, that

$$\Phi_{0u} = \begin{pmatrix} \phi & 0 \\ u & \phi' \end{pmatrix}, \qquad \Phi_{2u} = \begin{pmatrix} \phi' & 0 \\ u & \phi \end{pmatrix}$$

are solution matrices of (1). The general term of the golden sequence generated by Φ_{0u} is easily shown to be

$$\Phi_{0u}^n = \begin{pmatrix} \phi^n & 0 \\ F_n u & \phi'^n \end{pmatrix},$$

where

$$F_1 = 1, \qquad F_2 = 1, \qquad F_3 = 2, \qquad F_4 = 3, \cdots, \qquad F_n = F_{n-1}+F_{n-2}, \cdots,$$

the Fibonacci sequence. (For elementary properties of the latter, cf. [1, 4].)

Case 2. $y \neq 0$.

(a) If $u = 0$, Eqs. (3.1) and (3.4) reduce to the Fibonacci equation, and Eq. (3.3) implies $x+v = 1$. The situation is similar to the one in Case 1(b). We will return, however, to the matrix Φ_{0y} in Section 4.

(b) Suppose $u \neq 0$. Equation (3.3) implies $x = 1-v$ (consistent with Eq. (3.2)). Substitution for x in Eq. (3.1) results in

$$(1-v)^2-(1-v)-1+yu = 0,$$

which after simplification reduces to $v^2-v-1+yu = 0$, consistent with Eq. (3.4). Thus, the assumptions $y \neq 0$, $u \neq 0$ reduce the system (3.1) to (3.4) to the following equivalent system:

$$(4.1)\qquad v = \tfrac{1}{2}(1 \pm \sqrt{5-4yu})$$

$$(4.2)\qquad x = 1-v,$$

where $y \neq 0$, $u \neq 0$, are otherwise arbitrary. It is in this form of the equations that we will systematically investigate various sets of solutions of (1) in the next section.

3. EXAMPLES OF GOLDEN SEQUENCES

Example 1: If we limit y, u to positive integer values in (4.1), then there is a unique pair which keeps the radicand positive: $y = u = 1$. In this case, we have two sets of solutions:

$$x = 0, \quad v = 1, \quad y = 1, \quad u = 1; \quad \text{and} \quad x = 1, \quad v = 0, \quad y = 1, \quad u = 1.$$

The latter set results in the so-called "Q-matrix" [3, 4]:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and the corresponding golden sequence

$$I, Q, Q^2, \cdots, Q^n, \cdots,$$

where [3, 4]

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Example 2: As we observed in the previous section, and as we may now corroborate from Eq. (4.1), the pair $y = u = 0$, results in the matrix Φ_0, and the corresponding golden sequence

$$I, \Phi_0, \Phi_0^2, \cdots, \Phi_0^n, \cdots,$$

where

$$\Phi_0^n = \begin{pmatrix} \phi^n & 0 \\ 0 & \phi'^n \end{pmatrix}.$$

A natural question is whether or not Q and Φ_0 are related. A calculation shows that the characteristic equation for Q is

(5) $$\lambda^2 - \lambda - 1 = 0,$$

(the Fibonacci equation), the roots of which are ϕ and ϕ', the diagonal entries of Φ_0.

Thus, Eq. (5) is the characteristic equation for both Q and Φ_0, and by the Cayley-Hamilton theorem, each of these matrices satisfies this equation. A comparison of Eqs. (5) and (0) shows that we have in fact a characterization for all matrices which give rise to golden sequences:

Theorem 1. A necessary and sufficient condition for a matrix A to be a generator of a golden sequence is that its characteristic equation is the Fibonacci equation.

Since our hypotheses on the matrix A imply that the characteristic equation is, in fact, the minimal equation for A, we have

Corollary 1. Any two matrix generators of non-trivial golden sequences of matrices are similar.

Corollary 2. Q is similar to Φ_0; i.e., there exists a non-singular matrix T such that

$$Q = T\Phi_0T^{-1},$$

where the columns of T are eigenvectors of Q corresponding respectively to the eigenvalues ϕ and ϕ'.

In what follows (see Section 4) we will require the matrix T. A straight-forward computation shows that

$$T = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix};$$

this is easily checked by observing that, in fact, $QT = T\Phi_0$.

From Corollary 1, we infer that

$$Q^n = T\Phi_0^nT^{-1},$$

and hence, that Q^n is similar to Φ_0^n. Hence,

$$\det(Q^n) = \det(\Phi_0^n), \qquad \operatorname{trace}(Q^n) = \operatorname{trace}(\Phi_0^n),$$

and we have our first pair of Fibonacci identities:

Corollary 3. (i) $$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

(ii) $$F_{n+1} + F_{n-1} = \phi^n + \phi'^n,$$

$n = 1, 2, 3, \cdots$.

Remark 1: Since $L_n = F_{n+1} + F_{n-1}$ [3], where L_n is the general term of the Lucas sequence [3]

(6) $$1, 3, 4, 7, 11, \cdots,$$

it would appear that line (ii) in Corollary 3 establishes a proof for the Binet formula [3] : $L_n = \phi^n + \phi'^n$. However, the formula, $L_n = F_{n+1} + F_n$ is generally established from the principal Binet formula [3] :

$$F_n = (\phi^n + \phi'^n)/(\phi - \phi').$$

Although we have enough machinery at this point to establish the latter, the proof is not an immediate consequence of the similarity invariants, "trace" and "determinant" (which we would like to limit ourselves to in this section); thus, we defer this proof until Section 4. We do, however, establish the formula: $L_n = F_{n+1} + F_n$ in the next example, within our own framework.

Remark 2: Motivated from the general term, Q^n, in Example 1, where

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

it is natural to inquire as to whether the sequence with general term

$$P^n = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix},$$

is golden. However, since for $n = 1$, and setting $P = P^1$,

$$P = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix},$$

we see that P does not satisfy the Fibonacci equation; thus we conclude by Theorem 1 that P^n is not a golden sequence. Nevertheless, we will show in the next example that the Lucas numbers, (6), do, in fact, enter the picture in a natural way.

Example 3: Referring again to Eqs. (4.1), (4.2), we take $y = 1$, $u = 5/4$; then $v = ½$, $x = ½$, and we obtain the sequence generator

$$H = \begin{pmatrix} 1/2 & 1 \\ 5/4 & 1/2 \end{pmatrix},$$

and the corresponding golden sequence

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{5}{4} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\cdot 3 & 1 \\ \frac{5}{4} & \frac{1}{2}\cdot 3 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\cdot 4 & 2 \\ \frac{5}{4}\cdot 2 & \frac{1}{2}\cdot 4 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\cdot 7 & 3 \\ \frac{5}{4}\cdot 3 & \frac{1}{2}\cdot 7 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\cdot 11 & 5 \\ \frac{5}{4}\cdot 5 & \frac{1}{2}\cdot 11 \end{pmatrix}, \cdots,$$

where the general term is easily shown to be

$$H^n = \begin{pmatrix} \frac{1}{2}L_n & F_n \\ \frac{5}{4}F_n & \frac{1}{2}L_n \end{pmatrix}.$$

Similarity of Q^n (see Example 1) with H^n implies, by the invariance of trace, that

$$(7) \qquad L_n = F_{n+1} + F_{n-1},$$

and by determinant invariance, that

$$\frac{1}{4}L_n^2 - \frac{5}{4}F_n^2 = F_{n+1}F_{n-1} - F_n^2,$$

which after simplification becomes

$$(8) \qquad L_n^2 = 4F_{n+1}F_{n-1} + F_n^2.$$

Whereas, similarity of Φ^n with H^n implies

$$(9) \qquad L_n = \phi^n + \phi'^n \qquad \text{(Binet)},$$

$$(10) \qquad L_n^2 - 5F_n^2 = 4(-1)^n.$$

Example 4: In (4.1), take $y = 1$, $u = -1$; then one set of solutions is $v = 2$, $x = -1$, and we obtain the matrix

$$F = \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix}.$$

The general term of the corresponding golden sequence is easily seen to be

$$F^n = \begin{pmatrix} -F_{n-2} & F_n \\ -F_n & F_{n+2} \end{pmatrix}.$$

Similarity with Φ^n gives

(11) $$F_{n+2} - F_{n-2} = \phi^n + \phi'^n$$

and

(12) $$F_n^2 - F_{n+2}F_{n-2} = (-1)^n .$$

NOTE: In what follows, we shall only use those similarity results which produce identities not already established.

Similarity with Q^n gives

(13) $$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2},$$

and

(14) $$F_{n+1}F_{n-1} + F_{n+2}F_{n-2} = 2F_n^2 .$$

Similarity with H^n gives

(15) $$L_n = F_{n+2} - F_{n-2}$$

and

$$\tfrac{1}{4}L_n^2 - \tfrac{5}{4}F_n^2 = F_n^2 - F_{n+2}F_{n-2},$$

which after simplification becomes

(16) $$L_n^2 = 9F_n^2 - 4F_{n+2}F_{n-2}.$$

Example 5: By taking $y = 1$, $u = -5$ in (4.1), we obtain $v = -2$, $x = 3$, and the generator

$$L = \begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix} .$$

The corresponding golden sequence is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ -5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & 2 \\ -5\cdot 2 & -3 \end{pmatrix}, \begin{pmatrix} 11 & 3 \\ -5\cdot 3 & -4 \end{pmatrix}, \cdots,$$

with general term

$$L^n = \begin{pmatrix} L_{n+1} & F_n \\ -5F_n & -L_{n-1} \end{pmatrix} .$$

Similarity with Q^n implies

(17) $$L_{n+1}L_{n-1} + F_{n+1}F_{n-1} = 6F_n^2 .$$

Similarity with Φ^n implies

(18) $$5F_n^2 - L_{n+1}L_{n-1} = (-1)^n .$$

Similarity with H^n gives

(19) $$L_n^2 + 4L_{n+1}L_{n-1} = 25F_n^2.$$

and similarity with F^n gives

(20) $$L_{n+1} - L_{n-1} = F_{n+2} - F_{n-2},$$

and

(21) $$L_{n+1}L_{n-1} - F_{n+2}F_{n-2} = 4F_n^2.$$

Remark 3: Although there appear to be infinitely many more golden sequences we could investigate, subject only to the constraining equations (4.1) and (4.2), and thus, a limitless supply of Fibonacci identities to discover (or, rediscover) via the similarity invariants, "trace" and "determinant," we switch our direction at this point.

In Section 4, we offer two final examples of generators of golden sequences, and compute their eigenvectors. With this new tool we will then establish Binet's formula for F_n in terms of ϕ, ϕ' and their powers, and the formulas [3] for ϕ and ϕ' in terms of F_n and L_n.

4. PROOFS OF SOME CLASSICAL FORMULAS

In (4.1) take $u = 0$, $y \neq 0$, but for the time being arbitrary. Then $v = \phi$, $x = \phi'$, and we have the matrix (cf. Section 1)

$$\Phi_{oy} = \begin{pmatrix} \phi & y \\ 0 & \phi' \end{pmatrix} .$$

Setting $\Phi_y = \Phi_{oy}$, one easily checks that we generate the golden sequence

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \phi & y \\ 0 & \phi' \end{pmatrix}, \begin{pmatrix} \phi^2 & y \\ 0 & \phi'^2 \end{pmatrix}, \begin{pmatrix} \phi^3 & 2y \\ 0 & \phi'^3 \end{pmatrix}, \cdots,$$

where the general term is easily seen to be

$$\Phi_y^n = \begin{pmatrix} \phi^n & F_n y \\ 0 & \phi'^n \end{pmatrix},$$

The eigenvectors, corresponding to the eigenvalue ϕ, are computed to be $\begin{pmatrix} a \\ 0 \end{pmatrix}$, $a \neq 0$; we single out the eigenvector corresponding to $a = 1$; while the eigenvectors corresponding to ϕ' are of the form

$$\begin{pmatrix} a \\ \frac{1}{y}(\phi' - \phi)a \end{pmatrix}, \quad a \neq 0.$$

Since $\phi' - \phi = -\sqrt{5}$ (see Section 2, or [3]), we take $y = \sqrt{5}$ (so that $\Phi_y = \Phi_{\sqrt{5}}$), and $a = 1$. Thus we have the two eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

corresponding to the eigenvalues ϕ and ϕ', respectively. Set

$$S = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then by Corollary 1,

$$\Phi_{\sqrt{5}} = S\Phi_0 S^{-1},$$

which implies that

$$\Phi^n_{\sqrt{5}} = S\Phi^n_0 S^{-1},$$

and hence, that

(22) $$S\Phi^n_{\sqrt{5}} = S\Phi^n_0.$$

We write out Eq. (22):

$$\begin{pmatrix} \phi^n & F_n\sqrt{5} \\ 0 & \phi'^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \phi'^n \end{pmatrix}.$$

Multiplying out gives

$$\begin{pmatrix} \phi^n & \phi^n - F_n\sqrt{5} \\ 0 & -\phi'^n \end{pmatrix} = \begin{pmatrix} \phi^n & \phi'^n \\ 0 & -\phi'^n \end{pmatrix},$$

which implies that

$$\phi^n - F_n\sqrt{5} = \phi'^n ;$$

or

(*) $$F_n = \frac{\phi^n - \phi'^n}{\phi - \phi'}, \quad \text{(Binet)}$$

For our final example, we will permit our generator matrix to be complex. In (4.1), take $y = ½$, $u = 3$; then take $v = (1 + i)/2$, so that $x = (1 - i)/2$, and we obtain the matrix

$$C = \begin{pmatrix} (1-i)/2 & 1/2 \\ 3 & (1+i)/2 \end{pmatrix}.$$

The corresponding golden sequence is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \tfrac{1}{2}(1-i) & \tfrac{1}{2} \\ 3 & \tfrac{1}{2}(1+i) \end{pmatrix}, \begin{pmatrix} \tfrac{1}{2}(3-i) & \tfrac{1}{2} \\ 3 & \tfrac{1}{2}(3+i) \end{pmatrix}, \begin{pmatrix} \tfrac{1}{2}(4-2i) & \tfrac{1}{2}\cdot 2 \\ 3\cdot 2 & \tfrac{1}{2}(4+2i) \end{pmatrix}, \cdots,$$

with general term

$$C^n = \begin{pmatrix} \tfrac{1}{2}(L_n - F_n i) & \tfrac{1}{2}F_n \\ 3F_n & \tfrac{1}{2}(L_n + F_n i) \end{pmatrix}.$$

Proceeding as in the previous example, we take as eigenvector corresponding to the eigenvalue ϕ, the vector

$$\begin{pmatrix} \tfrac{1}{3}[\phi - \tfrac{1}{2}(1+i)] \\ 1 \end{pmatrix};$$

and corresponding to ϕ', we take the eigenvector

$$\begin{pmatrix} \tfrac{1}{3}[\phi' - \tfrac{1}{2}(1+i)] \\ 1 \end{pmatrix}.$$

Setting

$$B = \begin{pmatrix} \tfrac{1}{3}[\phi - \tfrac{1}{2}(1+i)] & \tfrac{1}{3}[\phi' - \tfrac{1}{2}(1+i)] \\ 1 & 1 \end{pmatrix},$$

we have by Corollary 1 that

$$C^nB = B\Phi_0^n. \tag{23}$$

Performing the indicated multiplication in (23) results in the matrix equation

$$\begin{pmatrix} \tfrac{1}{6}[L_n - F_n i][\phi - \tfrac{1}{2}(1+i)] + \tfrac{1}{2}F_n & \tfrac{1}{6}[L_n - F_n i][\phi' - \tfrac{1}{2}(1+i)] + \tfrac{1}{2}F_n \\ F_n[\phi - \tfrac{1}{2}(1+i)] + \tfrac{1}{2}[L_n + F_n i] & F_n[\phi' - \tfrac{1}{2}(1+i)] + \tfrac{1}{2}[L_n + F_n i] \end{pmatrix}$$
$$= \begin{pmatrix} \phi^n/3[\phi - \tfrac{1}{2}(1+i)] & \phi'^n/3[\phi' - \tfrac{1}{2}(1+i)] \\ \phi^n & \phi'^n \end{pmatrix}.$$

(a) Equating the corresponding entries in the second row, first column, and simplifying gives

$$L_n = 2\phi^n + (1-2\phi)F_n.$$

Solving for ϕ^n gives

$$\phi^n = \tfrac{1}{2}(L_n + \sqrt{5}F_n). \tag{24}$$

(b) Equating the corresponding terms in the first row, second column, and noting that these are identical to those obtained in (a) except that ϕ is replaced by ϕ', we have

$$L_n = 2\phi'^n + (1-2\phi')F_n;$$

or, solving for ϕ'^n, that

$$\phi'^n = \tfrac{1}{2}(L_n - \sqrt{5}F_n). \tag{25}$$

Remark 4: Equating the two remaining pairs of corresponding entries in the above matrix equation results in lines (24) and (25).

Remark 5: We chose the matrix

$$\Phi_{\sqrt{5}} = \begin{pmatrix} \phi & 5 \\ 0 & \phi' \end{pmatrix}$$

to establish the principal Binet formula (line (*)) because of the simplicity of the proof. It should be noted, however, that a proof within the framework of the Q-matrix [4] is also possible. Since the machinery has already been set up in Example 2, and because of the historical importance of this matrix, we give the proof. We have already established (similarity) that

$$Q^nT = T\Phi_0^n;$$

i.e., that

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}\begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}\begin{pmatrix} \phi^n & 0 \\ 0 & \phi'^n \end{pmatrix}.$$

Multiplying out gives

$$\begin{pmatrix} \phi F_{n+1} + F_n & F_{n+1} - \phi F_n \\ \phi F_n + F_{n-1} & F_n - \phi F_{n-1} \end{pmatrix} = \begin{pmatrix} \phi^{n+1} & \phi'^n \\ \phi^n & \phi'^{(n-1)} \end{pmatrix} .$$

Equating corresponding terms results in the following equivalent system of equations:

$$\begin{aligned} \phi F_{n+1} + F_n &= \phi^{n+1} \\ F_{n+1} - \phi F_n &= \phi'^n \\ \phi F_n + F_{n-1} &= \phi^n \\ F_n - \phi F_{n-1} &= \phi'^{(n-1)} . \end{aligned}$$

Solving the second equation for F_{n+1} and substituting this into the first equation, gives

$$\phi(\phi'^n + \phi F_n) + F_n = \phi^{n+1} .$$

Multiplying through by $-\phi'$ gives

$$\phi'^n + \phi F_n - \phi' F_n = \phi^n .$$

Finally, solving for F_n gives the desired result:

$$F_n = \frac{\phi^n - \phi'^n}{\phi - \phi'} .$$

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From (9a) and (9b), we obtain

(10a)
$$\sum_{n=-\infty}^{\infty} F_{(2k+1)n} J_n(x) = 0$$

and

(10b)
$$\sum_{n=-\infty}^{\infty} F_{(2k+1)n-1} J_n(x) = \exp\left(\frac{x}{2} L_{2k+1}\right) .$$

Equations (10a) and (10b) can be combined in the following equation, as may be shown by induction

(11)
$$\sum_{n=-\infty}^{\infty} F_{(2k+1)n+m} J_n(x) = F_m \exp\left(\frac{x}{2} L_{2k+1}\right) .$$

With $k = 0$ and $m = 1$, (11) becomes

$$\sum_{n=-\infty}^{\infty} F_{n+1} J_n(x) = \exp\frac{x}{2} .$$

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