**EXAMPLES:** $F_6/F_7$ is the 31st term in the sequence for $n = 10$;

$F_7/F_8$ is the 43rd term in the sequence for $n = 11$;

and $F_4/F_9$ has the same position in the sequence for $n = 11$, as $F_4/F_5$ has in that for $n = 7$. This means that $F_4/F_9$ is the 12th term in the sequence for $n = 11$.

## REFERENCES


1. Krishnaswami Alladi, "A Farey Sequence of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 13, No. 1 (Feb. 1975), pp. 1–10.
2. ______________, "A Rapid Method to Form Farey Fibonacci Fractions," *The Fibonacci Quarterly,* Vol. 13, No. 1 (Feb. 1975), pp. 31–32.


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# ON CONSECUTIVE PRIMITIVE ROOTS


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The purpose of this note is to determine which positive integers have their primitive roots consecutive. Of course, if "consecutive primitive roots" is taken to include integers which have only one primitive root, then 2, 3, 4, and 6 would qualify with primitive roots 1, 2, 3, and 5, respectively. It will be shown that 5, with primitive roots 2 and 3, is the only positive integer which has its primitive roots (plural) consecutive. It is well known that the only positive integers $m$, greater than 4, which have primitive roots are of the form $p^n$ or $2p^n$, $n \geq 1$, $p$ an odd prime. Most of these can be eliminated by the first two theorems.

*Theorem 1.* If $m = 2p^n$ $(m > 6)$, $n \geq 1$, $p$ an odd prime, then the primitive roots are not consecutive.

*Proof.* Primitive roots must have inverses, and, consequently, must be relatively prime to the modulus. With $m > 6$, there will be at least two primitive roots. Therefore, there are at least two odd primitive roots and no even primitive roots; they are not consecutive.

*Theorem 2.* If $m = p^n$, $n \geq 2$, $p$ an odd prime, then the primitive roots are not consecutive.

*Proof.* For $n \geq 3$,

$$p < p^{n-2}(p-1)\phi(p-1) = \phi(\phi(p^n)).$$

This implies that multiples of $p$ occur within a span less than $\phi(\phi(p^n))$. Now, multiples of $p$ are not relatively prime to the modulus, and are, therefore, not primitive roots. Since there are $\phi(\phi(p^n))$ primitive roots, they cannot be consecutive. For n = 2, $\phi(\phi(p^2)) = (p-1)\phi(p-1)$. For $p > 3$, $\phi(p-1) \geq 2$, and so,

$$(p-1)\phi(p-1) \geq 2(p-1) = 2p-2 = p+p-2 > p.$$

The conclusion follows as in the case $n \geq 3$. For $m = 3^2$, the primitive roots are 2 and 5, and not consecutive.

*Lemma.* If $p$ is an odd prime greater than 5 and not equal to 7, 11, 13, 19, 31, 43, 61, then $2\sqrt{p-1} \leq \phi(p-1)$.

*Proof.* The conclusion is equivalent to $4(p-1) \leq [\phi(p-1)]^2$. Let $p - 1 = 2^a p_1^{a_1} \cdots p_n^{a_n}$, and suppose that $4(p-1) > [\phi(p-1)]^2$. Then,

$$(1) \qquad 2^{a+2}p_1^{a_1} \cdots p_n^{a_n} > 2^{2(a-1)}p_1^{2(a_1-1)} \cdots p_n^{2(a_n-1)}(p_1-1)^2 \cdots (p_n-1)^2.$$

If $p - 1 = 2^a$, then (1) reduces to $2^{a+2} > 2^{2(a-1)}$. This implies that $16 > 2^a$, or $a < 4$. Thus, $p = 3$ or $5$.

Otherwise, (1) reduces to

$$(2) \qquad 8 > 2^{a-1}p_1^{a_1-2}(p_1-1)^2 p_2^{a_2-2}(p_2-1)^2 \cdots p_n^{a_n-2}(p_n-1)^2.$$

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