ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-269 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas

The sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, defined by

$$a_n = \sum_{k=0}^{[n/3]} \binom{n-2k}{k} \quad (n \geq 1) \quad \text{and} \quad b_{2n} = \sum_{k=0}^{[n/2]} \left[\begin{matrix} n-k \\ 2k \end{matrix}\right], \quad b_{2n+1} = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \left[\begin{matrix} n-k \\ 2k+1 \end{matrix}\right], \quad (n \geq 0)$$

are obtained as diagonal sums from Pascal's triangle and from a similar triangular array of numbers formed by the coefficients of powers of x in the expansion of $(x^2+x+1)^n$, respectively. (More precisely, $\left[\begin{matrix} n \\ k \end{matrix}\right]$ is the coefficient of x^k in $(x^2+x+1)^n$.) Verify that $a_n = b_{n-1} + b_n$ for each $n = 1, 2, \cdots$.

H-270 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S \equiv \sum_{a,b,c} \frac{x^a y^b z^c}{(b+c-a)!\,(c+a-b)!\,(a+b-c)!},$$

where the summation is over all non-negative a, b, c such that

$$a \leq b+c, \quad b \leq a+b, \quad c \leq a+b.$$

B-271 Proposed by R. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Define the binary dual, D, as follows:

$$D = \left\{ t \mid t = \prod_{i=0}^{n} (a_i + 2i); \quad a_i \in \{0,1\}; \quad n \geq 0 \right\}.$$

Let $\bar{D}$ denote the complement of D*. Form a sequence, $\{S_n\}_{n=1}^{\infty}$, by arranging $\bar{D}$ in increasing order. Find a formula for S_n.

NOTE: The elements of D result from interchanging t and x in a binary number.

*With respect to the set of positive integers.

SOLUTIONS

UNITY WITH FIBONACCI

H-247 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that for each Fibonacci number, F_r, there exist an infinite number of positive nonsquare integers, D, such that

$$F_{r+s}^2 - F_r^2 D = 1.$$

Solution by the Proposer.

(1) $$F_{r+s}^2 - 1 = (F_{r+s}+1)(F_{r+s}-1) = F_r^2 D.$$

It is assumed as well known that every positive integer is a divisor of an infinite number of Fibonacci numbers. Also, a congruence table modulo F_r^2 of the Fibonacci numbers is not only periodic but each period starts 1, 1, ⋯. Hence there are at least two infinite chains of Fibonacci numbers which are congruent to 1 modulo F_r^2 thus making D in (1) an integer. Since the difference of two positive squares is never one, D in (1) is a non-square integer.

Example: $$F_r = F_4 = 3$$

Congruence Table Modulo 9

1	1	2	3	5	8	4	3	7	1	8	0
8	8	7	6	4	1	5	6	2	8	1	0

F_{r+s} can be chosen as any one of the Fibonacci numbers

$F_{24n+1}, F_{24n+2}, F_{24n+6}, F_{24n+10}, F_{24n+11}, F_{24n+13}, F_{24n+14}, F_{24n+18}, F_{24n+22}, F_{24n+23}$.

$$\begin{aligned} F_6 &= 8, & D &= 7\\ F_{10} &= 55, & D &= 336\\ F_{11} &= 89, & D &= 880\\ F_{13} &= 233, & D &= 6032\\ F_{18} &= 2584, & D &= 741895\\ F_{22} &= 17711, & D &= 34853280\\ F_{23} &= 28657, & D &= 91247072 \end{aligned}$$

Also solved by P. Bruckman.

THE VERY EXISTENCE

H-248 Proposed by F. D. Parker, St. Lawrence University, New York.

A well known identity for the Fibonacci numbers is

$$F_n^2 - F_{n-1}F_{n+1} = -(-1)^n$$

and a less well known identity for the Lucas numbers is

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n.$$

More generally, if a sequence $\{y_0, y_1, \cdots\}$ satisfies the equation $y_n = y_{n-1} + y_{n-2}$, and if y_0 and y_1 are integers, then there exists an integer N such that

$$y_n^2 - y_{n-1}y_{n+1} = N(-1)^n.$$

Prove this statement and show that N cannot be of the form $4k+2$, and show that $4N$ terminates in 0, 4, or 6.

Solution by G. Berzsenyi, Lamar University, Beaumont, Texas.

By use of the above identity for Fibonacci numbers and the well known relation

$$y_n = F_{n-1}y_0 + F_n y_1,$$

we first establish that

$$N = y_0^2 + y_0y_1 - y_1^2.$$

Indeed,

$$\begin{aligned} y_n^2 - y_{n-1}y_{n+1} &= (F_{n-1}y_0 + F_ny_1)^2 - (F_{n-2}y_0 + F_{n-1}y_1)(F_ny_0 + F_{n+1}y_1)\\ &= (F_{n-1}^2 - F_{n-2}F_n)y_0^2 + (F_n^2 - F_{n-1}F_{n+1})y_1^2 + (F_{n-1}^2 - F_{n-2}F_n)y_0y_1\\ &= -(-1)^{n-1}y_0^2 - (-1)^n y_1^2 - (-1)^{n-1}y_0y_1 = (y_0^2 + y_0y_1 - y_1^2)(-1)^n. \end{aligned}$$

It is easy to see that N is odd unless both y_0 and y_1 are even, in which case it is a multiple of 4. Thus it cannot be of the form $4k+2$.

By a case-by-case examination of the congruences of y_0 and y_1 (mod 5) one can also establish that

$$y_0^2 + y_0y_1 - y_1^2 = 0,\ 1 \text{ or } 4 \pmod 5.$$

Therefore, there exists no integer t such that

$$y_0^2 + y_0y_1 - y_1^2 = 5t+2 \qquad \text{or} \qquad y_0^2 + y_0y_1 - y_1^2 = 5t+3.$$

Consequently, $4N$ is not of the following forms

$$4(5t+2) = 20t+8 = 10(2t)+8 \qquad \text{and} \qquad 4(5t+3) = 20t+12 = 10(2t+1)+2,$$

i.e., the last digit of $4N$ is 0, 4 or 6.

Also solved by A. Shannon, J. Biggs, J. Howell, C. B. A. Peck, P. Bruckman, J. Ivie, and the Proposer.

FOLK-LAURIN

H-249 Proposed by F. D. Parker, St. Lawrence University, Canton, New York.

Find an explicit formula for the coefficients of the Maclaurin series for

$$\frac{b_0 + b_1x + \cdots + b_kx^k}{1+\alpha x+\beta x^2}.$$

Since two quite different solutions were offered by the Proposer and by P. Bruckman, we present both solutions.

Solution by the Proposer.

We first get a Maclaurin series for the reciprocal of $1+\alpha x+\beta x^2$. Since we require the values for a_n for which

$$1 = (1+\alpha x+\beta x^2)(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$$

we say that $a_0 = 1$, $a_1 = -\alpha$, and

$$a_n + \alpha a_{n-1} + \beta a_{n-2} = 0.$$

Thus the coefficients a_n satisfy a second-order difference equation which is both linear and homogeneous. The general solution is

$$a_n = c_1x_1^n + c_2x_2^n,$$

where x_1 and x_2 are solutions to the equation

$$1+\alpha x+\beta x^2 = 0.$$

Since $a_0 = 1$ and $a_1 = -\alpha$, we can evaluate the constants c_1 and c_2 to get

$$a_n = \frac{2x_2+x_1}{x_2-x_1}x_1^n + \frac{2x_1+x_2}{x_1-x_2}x_2^n.$$

Thus, we have

$$\frac{b_0 + b_1x + \cdots + b_kx^k}{1+\alpha x+\beta x^2} = d_0 + d_1x + \cdots + d_nx^n + \cdots,$$

and

$$d_n = \sum_{i=0}^{r} a_{n-i}b_i,$$

where $r = \min(n,k)$.

If the roots x_1 and x_2 are equal, then a_n takes the easier form of $a_n = (1+3n)x_1^n$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

The following little-known determinant theorem was brought to my attention by Dr. Furio Alberti, U.I.C.C.: If

$$\frac{A_0 + A_1x + A_2x^2 + \cdots}{B_0 + B_1x + B_2x^2 + \cdots} = C_0 + C_1x + C_2x^2 + \cdots,$$

and $B_0 \neq 0$, *then*

$$C_n = \frac{(-1)^n}{B_0^{n+1}} = \begin{vmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} & A_n \\ B_0 & B_1 & B_2 & \cdots & B_{n-1} & B_n \\ 0 & B_0 & B_1 & \cdots & B_{n-2} & B_{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_0 & B_1 \end{vmatrix}_{(n+1)X(n+1)}$$

In our particular problem, $A_n = b_n$, $n = 0, 1, \cdots$, with $b_n = 0$ if $n > k$; also, $B_0 = 1$, $B_1 = \alpha$, $B_2 = \beta$, $B_n \neq 0$ if $n \geqslant 3$. Therefore,

$$c_n = (-1)^n \begin{vmatrix} b_0 & b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ 1 & \alpha & \beta & 0 & \cdots & 0 & 0 \\ 0 & 1 & \alpha & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \alpha \end{vmatrix}_{(n+1)X(n+1)}$$

GROWTH RATE

H-250 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that if

$$A(n) = F_{n+1} + B(n)F_n = C(n) \qquad (n = 0, 1, 2, \cdots),$$

where the F_n are the Fibonacci numbers and $A(n)$, $B(n)$, $C(n)$ are polynomials, then

$$A(n) \equiv B(n) \equiv C(n) \equiv 0.$$

Solution by the Proposer.

We shall prove the following more general result.

Theorem. Let r be a fixed positive integer and $\alpha > 1$ a fixed real number. Assume that

$$(*) \qquad A_0(n)\alpha^{rn} + A_1(n)\alpha^{(r-1)n} + \cdots + A_r(n) = 0 \qquad (n = 0, 1, 2, \cdots),$$

where the $A_j(n)$ are polynomials in n. Then

$$A_0(n) \equiv \cdots \equiv A_r(n) \equiv 0.$$

Proof. We may assume that $A_0(n) \not\equiv 0$. Put

$$A_0(n) = \sum_{j=0}^{k} a_jx^j, \qquad a_k \neq 0.$$

Divide both sides of (*) by $n^k\alpha^{rn}$ and let $n \to \infty$. This gives $a_k = 0$, thus proving the theorem.

In the given equation

$$A(n)F_{n+1} + B(n)F_n = C(n),$$

put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \qquad \beta = \tfrac{1}{2}(1 - \sqrt{5}).$$

Then

$$A(n)(\alpha^{2n+2} - 1) + B(n)(\alpha^{2n+1} - \alpha) = C(n)(\alpha - \beta)\alpha^n,$$

so that

$$(\alpha^2A(n) + \alpha B(n))\alpha^{2n} - (\alpha - \beta)C(n)\alpha^n - (A(n) + \alpha B(n)) = 0.$$

Hence by the theorem

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