FIBONACCI NOTES
5. ZERO-ONE SEQUENCES AGAIN

L. CARLITZ*
Duke University, Durham, North Carolina 27706

1. The point of view of the present paper is somewhat different from that in [1]. We shall now consider the following problem.

Let $f(m,n,r,s)$ denote the number of zero-one sequences of length $m+n$:

$$(a_1, a_2, \cdots, a_{m+n}) \qquad (a_i = 0 \text{ or } 1) \tag{1.1}$$

with m zeros, n ones, r occurrences of (00) and s occurrences of (11).

Examples.

I. $m = 3,\ n = 2,\ r = 1,\ s = 0$

(0 0 1 0 0)
(1 0 1 0 0)
(1 0 0 1 0) $\qquad f(3,2,1,0) = 4$
(0 1 0 0 1)

II. $m = 4,\ n = 2,\ r = 2,\ s = 1$

(0 0 1 1 0 0)
(0 0 0 1 1 0) $\qquad f(4,2,2,1) = 3$
(0 1 1 0 0 0)

III. $m = 4,\ n = 2,\ r = 1,\ s = 1 \qquad f(4,2,1,1) = 0.$

In order to evaluate $f(m,n,r,s)$ it is convenient to define $f_j(m,n,r,s)$, the number of sequences (1.1) with m zeros, n ones, r occurrences of (00), s occurrences of (11) and with $a_1 = j$, where $j = 0$ or 1. It follows immediately from the definition that $f_j(m,n,r,s)$ satisfies the following recurrences.

$$\begin{cases} f_0(m,n,r,s) = f_0(m-1, n, r-1, s) + f_1(m-1, n, r, s) \\ f_1(m,n,r,s) = f_0(m, n-1, r, s) + f_1(m, n-1, r, s-1), \end{cases} \tag{1.2}$$

where $m \geqslant 1,\ n \geqslant 1$ and it is understood that

$$f_j(m,n,r,s) = 0 \qquad (j = 0 \text{ or } 1)$$

if any of the parameters m,n,r,s is negative. We also take

$$\begin{cases} f_0(1,0,0,0) = f_1(0,1,0,0) = 1 \\ f_0(1,0,r,s) = f_1(0,1,r,s) = 0 \qquad (r+s > 0) \end{cases} \tag{1.3}$$

and

$$f_1(0,0,r,s) = 0 \qquad (j = 0 \text{ or } 1) \tag{1.4}$$

for all $r,s \geqslant 0$.

Now put

$$F_j \equiv F_j(x,y,u,v) = \sum_{m,n,r,s=0}^{\infty} f_j(m,n,r,s)x^m y^n u^r v^s \tag{1.5}$$

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and

(1.6) $$F \equiv F(x,y,u,v) = F_0(x,y,u,v) + F_1(x,y,u,v)\ .$$

It follows from (1.2), (1.3), (1.4) and (1.5) that

$$\begin{cases} F_0(x,y,u,v) = x + xuF_0(x,y,u,v) + xF_1(x,y,u,v) \\ F_1(x,y,u,v) = y + yF_0(x,y,u,v) + yvF_1(x,y,u,v), \end{cases}$$

or more compactly

(1.7) $$\begin{cases} (1-xu)F_0 - xF_1 = x \\ -yF_0 + (1-yv)F_1 = y\ . \end{cases}$$

Solving this system of equations we get

(1.8) $$\begin{cases} F_0 = \dfrac{x(1-yv)+xy}{(1-xu)(1-yv)-xy} \\ F_1 = \dfrac{xy+y(1-xu)}{(1-xu)(1-yv)-xy}\ . \end{cases}$$

Therefore, by (1.6),

(1.9) $$F(x,y,u,v) = \frac{x+y+2xy-xy(u+v)}{(1-xu)(1-yv)-xy}\ .$$

In the next place, we have

$$\frac{1}{(1-xu)(1-yv)-xy} = \sum_{k=0}^{\infty} \frac{(xy)^k}{(1-xu)^{k+1}(1-yv)^{k+1}} = \sum_{k=0}^{\infty}(xy)^k \sum_{r,s=0}^{\infty}\binom{r+k}{k}\binom{s+k}{k}(xu)^r(yv)^s$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} u^{m-k}v^{n-k}\ .$$

It then follows from (1.9) that

$$F = \sum_{m,n,k}^{\infty}\Big\{\binom{m-1}{k}\binom{n}{k}x^my^nu^{m-k-1}v^{n-k} + \binom{m}{k}\binom{n-1}{k}x^my^nu^{m-k}v^{n-k-1}$$

$$+2\binom{m-1}{k}\binom{n-1}{k}x^my^nu^{m-k-1}v^{n-k-1} - \binom{m-1}{k}\binom{n-1}{k}x^my^nu^{m-k}v^{n-k-1}$$

(1.10) $$-\binom{m-1}{k}\binom{n-1}{k}x^my^nu^{m-k-1}v^{n-k}\Big\}$$

$$= \sum_{m,n,k}^{\infty}\Big\{\binom{m-1}{k}\binom{n-1}{k-1}x^my^nu^{m-k-1}v^{n-k} + \binom{m-1}{k-1}\binom{n-1}{k}x^my^nu^{m-k}v^{n-k-1}$$

$$+2\binom{m-1}{k}\binom{n-1}{k}x^my^nu^{m-k-1}v^{n-k-1}\Big\}\ .$$

Since

$$F = \sum_{m,n,r,s=0}^{\infty} f(m,n,r,s)x^my^nu^rv^s\ ,$$

it follows from (1.10) that

(1.11) $$f(m,n,r,s) = \begin{cases} 2\binom{m-1}{r}\binom{n-1}{s} & (m-r=n-s) \\ \binom{m-1}{r}\binom{n-1}{s} & (m-r=n-s\pm 1) \\ 0 & \text{(otherwise)}. \end{cases}$$

This holds for all $m,n \geq 0$, except $m = n = 0$. If m (or n) = 0 clearly $f(m,n,r,s) = 0$ unless r (or s) = 0.

For example, (1.11) gives

$$f(3,2,1,0) = 2\binom{2}{1}\binom{1}{0} = 4$$

$$f(4,2,2,1) = \binom{3}{2}\binom{1}{1} = 3$$

$$f(4,2,1,1) = 0,$$

in agreement with the worked examples.

We may now state the following

Theorem 1. The enumerant $f(m,n,r,s)$ is evaluated by (1.11).

The simplicity of this result suggests that one may be able to find a direct combinatorial proof.

2. We now examine several special cases. First, for $x = y$, (1.9) becomes

$$F(x,x,u,v) = \frac{2x + 2x^2 - x^2(u+v)}{(1-xu)(1-xv) - x^2}. \tag{2.1}$$

Put

$$f(n,r,s) = \sum_{j+k=n}^{\infty} f(j,k,r,s), \tag{2.2}$$

so that $f(n,r,s)$ is the number of zero-one sequences of length n with r occurrences of (00) and s occurrences of (11). To evaluate $f(n,r,s)$ we make use of (1.11).

It is clear from (1.11) that the only values of j,k in (2.2) that we need consider are those satisfying

$$\begin{cases} j+k = n \\ j-k = r-s+(0,\ 1 \text{ or } -1). \end{cases} \tag{2.3}$$

Thus, for example, if

$$\begin{cases} j+k = n \\ j-k = r-s, \end{cases} \tag{2.4}$$

we must have

$$\begin{cases} n \equiv r+s \pmod 2 \\ n \geq |r-s|. \end{cases} \tag{2.5}$$

If (2.5) is satisfied it follows that

$$f(n,r,s) = 2\binom{\frac{1}{2}(n+r-s)-1}{r}\binom{\frac{1}{2}(n-r+s)-1}{s} \tag{2.6}$$

provided at least one of the numerators is non-negative.

Similarly, if

$$\begin{cases} j+k = n \\ j-k = r-s+1 \end{cases}, \tag{2.7}$$

we must have

$$\begin{cases} n \equiv r+s+1 \pmod 2 \\ n \geq |r-s+1| \end{cases} \tag{2.8}$$

and we get

$$f(n,r,s) = \binom{\frac{1}{2}(n+r-s+1)-1}{r}\binom{\frac{1}{2}(n-r+s-1)-1}{s}, \tag{2.9}$$

provided at least one numerant is non-negative.

Finally, if

$$\begin{cases} j+k = n \\ j-k = r-s-1, \end{cases} \tag{2.10}$$

we must have

(2.11)
$$\begin{cases} n \equiv r+s+1 \pmod 2 \\ n \geqslant r-s-1 \end{cases}$$

and we get

(2.12)
$$f(n,r,s) = \binom{\tfrac{1}{2}(n+r-s-1)-1}{r}\binom{\tfrac{1}{2}(n-r+s+1)-1}{s},$$

provided at least one numerant is non-negative.

In all other cases

(2.13)
$$f(n,r,s) = 0.$$

We may state

Theorem 2. The enumerant $f(n,r,s)$ defined by (2.2) is evaluated by (2.6), (2.9), (2.12) and (2.13).

3. We next take $u = v$ in (19) so that

(3.1)
$$F(x,y,u,u) = \frac{x+y+2xy-2xyu^2}{(1-xu)(1-yu)-xy}.$$

Define

(3.2)
$$g(m,n,t) = \sum_{r+s=t} f(m,n,r,s),$$

so that $g(m,n,t)$ is the number of zero-one sequences with m zeros, n ones and t occurrences of either (00) or (11). As in the previous case we need only consider

(3.3)
$$\begin{cases} r+s = t \\ m-n = r-s+(0,\ 1 \text{ or } -1). \end{cases}$$

We get the following results:

(3.4)
$$g(m,n,t) = 2\binom{m-1}{\tfrac{1}{2}(m-n+t)}\binom{n-1}{\tfrac{1}{2}(-m+n+t)}$$

provided

(3.5)
$$\begin{cases} m+n \equiv t \pmod 2 \\ t \geqslant |m-n| \end{cases};$$

(3.6)
$$g(m,n,t) = \binom{m-1}{\tfrac{1}{2}(m-n+t+1)}\binom{n-1}{\tfrac{1}{2}(-m+n+t-1)}$$

provided

(3.7)
$$\begin{cases} m+n \equiv t+1 \pmod 2 \\ t \geqslant |m-n+1| \end{cases};$$

(3.8)
$$g(m,n,t) = \binom{m-1}{\tfrac{1}{2}(m-n+t-1)}\binom{n-1}{\tfrac{1}{2}(-m+n+t+1)}$$

provided

(3.9)
$$\begin{cases} m+n \equiv t+1 \pmod 2 \\ t \geqslant |m-n-1| \end{cases};$$

in all other cases

(3.10)
$$g(m,n,t) = 0.$$

We may state

Theorem 3. The enumerant $g(m,n,t)$, defined by (3.2), is evaluated by (3.4), (3.7), (3.9) and (3.10).

4. For $x = y,\ u = v$, (1.9) reduces to

(4.1)
$$F(x,x,u,u) = \frac{2x+2x^2-2x^2u^2}{(1-xu)^2-x^2}.$$

Thus

$$F(x,x,u,u) = \frac{2x(1-x(u-1))}{(1-x(u+1))(1-x(u-1))} = \frac{2x}{1-x(u+1)} = 2\sum_{n=1}^{\infty} x^n(u+1)^{n-1} = 2\sum_{n=1}^{\infty} x^n \sum_{t=0}^{n-1}\binom{n-1}{t}u^t.$$

Hence if we put

(4.2)
$$h(n,t) = \sum_{\substack{j+k=n\\ r+s=t}} f(j,k,r,s),$$

it follows that

(4.3)
$$h(n,t) = 2\binom{n-1}{t} \qquad (0 \leq t < n).$$

The enumerant $h(n,t)$ can be described as the number of zero-one sequences of length n with t occurrences of either (00) or (11).

We may state

Theorem 4. The enumerant $h(n,t)$ defined by (4.2) is evaluated by (4.3).

This result can be proved by a combinatorial argument in the following way. Let the symbol x denote any doublet —either (00) or (11). Thus we are enumerating sequences of length $n - t$:

(4.4)
$$(a_1, a_2, \cdots, a_{n-t}),$$

where each a_i is equal to 0, 1 or x. Consecutive zeros and ones are ruled out; also if 0 is followed by x, then x stands for (11), while if 1 is followed by x, then x stands for (00). Thus we can describe the sequence (4.4) in the following way. Assume it begins with 0 or (00). Then we have a subsequence (0101 ⋯) of length r_0, followed by a subsequence $(xx \cdots)$ of length s_1, where the x's denote doublets of the same kind; this is followed by a subsequence of length r_1 which is either of the type (0101 ⋯) or (1010 ⋯) depending on the x, and so on. By the subsequence (xxx), for example, we understand (0000) or (1111). Thus, for the sequence,

$$\big(010(111)01(00)(11)\big)$$

we have $r_0 = 3,\ s_1 = 2,\ r_1 = 2,\ s_2 = 1,\ r_2 = 0,\ s_3 = 1,\ r_3 = 0,\ t = 4.$

Hence

(4.5)
$$h(n,t) = 2\sum 1,$$

where the summation is over non-negative $r_0, r_1, \cdots, r_k$ and positive $s_1, \cdots, s_k$ such that

(4.6)
$$\begin{cases} r_0 + r_1 + \cdots + r_k + s_1 + \cdots + s_k = n - k \\ s_1 + \cdots + s_k = t \qquad (k = 0, 1, 2, \cdots). \end{cases}$$

For $t = 0$ there is nothing to prove so we assume $t > 0$. Since

$$\#\left\{\begin{matrix} r_0 + r_1 + \cdots + r_k = n - k - t \\ r_i \geq 0 \end{matrix}\right\} = \binom{n-t}{k}$$

and

$$\#\left\{\begin{matrix} s_1 + \cdots + s_k = t \\ s_i > 0 \end{matrix}\right\} = \#\left\{\begin{matrix} s_1 + \cdots + s_k = t - k \\ s_i \geq 0 \end{matrix}\right\} = \binom{t-1}{k},$$

it follows from (4.5) and (4.6) that

$$h(n,t) = 2\sum_{k=1}^{t}\binom{n-t}{k}\binom{t-1}{k} = 2\sum_{k=0}^{t-1}\binom{n-t}{n-t-k-1}\binom{t-1}{k} = 2\binom{n-1}{n-t-1} = 2\binom{n-1}{t}.$$

5. For $v = 0$, (1.9) becomes

(5.1)
$$F(x,y,u,0) = \frac{x + y + 2xy - xyu}{1 - x(y + u)}.$$

The right-hand side of (5.1) is equal to

$$(x+y+2xy-xyu)\sum_{m=0}^{\infty} x^m \sum_{r=0}^{m}\binom{m}{r} y^{m-r}u^r = \sum_{m,r}\binom{m-1}{r} x^m y^{m-r-1}u^r + \sum_{m,r}\binom{m}{r} x^m y^{m-r+1}u^r$$

$$+2\sum_{m.r}\binom{m-1}{r} x^m y^{m-r}u^r - \sum_{m,r}\binom{m-1}{r-1} x^m y^{m-r+1}u^r$$

$$= \sum_{m,r}\binom{m-1}{r} x^m y^{m-r-1}u^r + \sum_{m,r}\binom{m-1}{r} x^m y^{m-r+1}u^r + 2\sum_{m,r}\binom{m-1}{r} x^m y^{m-r}u^r .$$

Since

$$F(x,y,u,0) = \sum_{m,n,r} f(m,n,r,0)x^m y^n u^r ,$$

it follows that

$$f(m,n,r,0) = \begin{cases} 2\binom{m-1}{r} & (m-n = r) \\ \binom{m-1}{r} & (m-n = r \pm 1). \end{cases}$$

If we take $u = 1$ in (5.1), we get

$$F(x,y,1,0) = \frac{x+y+xy}{1-x(y+1)} . \tag{5.2}$$

The RHS of (5.2) is equal to

$$(x+y+xy)\sum_{m=0}^{\infty} x^m \sum_{n=0}^{m}\binom{m}{n} y^n = \sum_{m,n}\left\{\binom{m-1}{n} + \binom{m}{n-1} + \binom{m-1}{n-1}\right\} x^m y^n = \sum_{m,n}\binom{m+1}{n} x^m y^n .$$

Hence

$$\sum_{r=0}^{m} f(m,n,r,0) = \binom{m+1}{n}. \tag{5.3}$$

Finally, for $x = y$, (5.2) reduces to

$$F(x,x,1,0) = \frac{2x+x^2}{1-x-x^2} = (2+x)\sum_{n=1}^{\infty} F_n x^n = \sum_{n=1}^{\infty} F_{n+2}x^n ,$$

where F_{n+2} is a Fibonacci number in the usual notation. It follows from (5.3) that

$$\sum_{j+k=n}\sum_{r=0}^{j} f(j,k,r,0) = F_{n+2} . \tag{5.4}$$

Clearly

$$\sum_{r=0}^{m} f(m,n,r,0)$$

is the number of zero-one sequences with m zeros, n ones and doublets (11) forbidden. Similarly

$$\sum_{j+k=n}\sum_{r=0}^{j} f(j,k,r,0)$$

is the number of zero-one sequences of length n with (11) forbidden. Thus (5.3) and (5.4) are familiar results.

6. Put

(6.1) $$F(x,y,u,v) = \sum_{m,n=0}^{\infty} F_{m,n}(u,v)x^m y^n ,$$

so that

(6.2) $$F_{m,n}(u,v) = \sum_{r=0}^{m} \sum_{s=0}^{n} f(m,n,r,s)u^r v^s ,$$

a polynomial in u and v. Thus (1.9) becomes

(6.3) $$\frac{x+y+(2-u-v)xy}{1-xu-yv-(1-uv)xy} = \sum_{m,n=0}^{\infty} F_{m,n}(u,v)x^m y^n .$$

It follows that

$$x+y+(2-u-v)xy = (1-xu-yv-(1-uv)xy) \sum_{m,n=0}^{\infty} F_{m,n}(u,v)x^m y^n .$$

Comparing coefficients, we get

(6.4) $$F_{m,n}(u,v) = uF_{m-1,n}(u,v) + vF_{m,n-1}(u,v) + (1-uv)F_{m-1,n-1}(u,v) \quad (m+n > 2).$$

It is evident from (6.3) that

(6.5) $$F_{m,n}(u,v) = F_{n,m}(v,u).$$

Also, taking $y = 0$, (6.3) reduces to

$$\frac{x}{1-xu} = \sum_{m=0}^{\infty} F_{m,0}(u,v)x^m .$$

Hence

(6.6) $$\begin{cases} F_{m,0}(u,v) = u^{m-1} & (m > 0) \\ F_{0,n}(u,v) = v^{n-1} & (n > 0) . \end{cases}$$

Since

$$F_{1,1}(u,v) - vF_{1,0}(u,v) - uF_{0,1}(u,v) = 2-u-v ,$$

it follows that

(6.7) $$F_{1,1}(u,v) = 2.$$

For $u = v = 1$, (6.3) becomes

$$\sum_{m,n=0}^{\infty} F_{m,n}(1,1)x^m y^n = \frac{x+y}{1-x-y} = \sum_{k}^{\infty} (x+y)^k = \sum_{m+n>0} \binom{m+n}{m} x^m y^n ,$$

so that

(6.8) $$F_{m,n}(1,1) = \binom{m+n}{m} \qquad (m+n > 0) .$$

By mans of (1.11) we can evaluate $F_{m,n}(u,v)$ explicitly, namely:

(6.9) $$\begin{aligned} F_{m,n}(u,v) = {} & 2 \sum_{s=0}^{n-1} \binom{m-1}{n-s-1} \binom{n-1}{s} u^{m-n+s} v^s \\ & + \sum_{s=0}^{n-1} \binom{m-1}{n-s} \binom{n-1}{s} u^{m-n+s+1} v^s \\ & + \sum_{s=0}^{n-2} \binom{m-1}{n-s-2} \binom{n-1}{s} u^{m-n+s-1} v^s \qquad (m \geqslant n \geqslant 1). \end{aligned}$$

For example, for $n = 1$,

$$F_{m,1}(u,v) = 2u^{m-1} + (m-1)u^m \qquad (m \geqslant 1),$$

so that

$$F_{1,n}(u,v) = 2v^{n-1} + (n-1)v^n \qquad (n \geqslant 1).$$

For $m = n$ we get

$$(6.10) \qquad F_{m,m}(u,v) = 2\sum_{r=0}^{m-1} \binom{m-1}{r}^2 (uv)^r + (u+v)\sum_{r=0}^{m-1} \binom{m-1}{r}\binom{m-1}{r+1} (uv)^r .$$

In connection with the recurrence (6.4), it may be of interest to point out that Stanton and Cowan [3] have discussed the recurrence

$$(6.11) \qquad g(n+1, r+1) = g(n, r+1) + g(n+1, r) + g(n, r)$$

subject to the initial conditions

$$g(n,0) = g(0,r) = 1 \qquad (n \geqslant 0,\ r \geqslant 0).$$

The more general recurrences

$$(6.12) \qquad A(n,r) = A(n-1, r-1) + q^n A(n, r-1) + q^r A(n-1, r)$$

and

$$(6.13) \qquad A(n, r) = A(n-1, r-1) + p^n A(n, r-1) + q^r A(n-1, r)$$

have been treated in [2].

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2. L. Carlitz, "Some q-analogs of Certain Combinatorial Numbers," *SIAM J. on Math. Analysis,* Vol. 4 (1973), pp. 433–446.
3. R. G. Stanton and D. D. Cowan, "Note on a 'Square' Functional Equation," *SIAM Review,* Vol. 12 (1972), pp. 277–279.

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If the relations (10), (11) and (12) are used, it can be shown that the much simpler expressions for the constants in the explicit solution (2) are indeed given by equations (9).

The generating function for the sequence P_r is defined by

$$(13) \qquad G = \sum_{r=0}^{\infty} x^r P_r = \sum_{r=0}^{\infty} [C_1(xR_1)^r + C_2(xR_2)^r + C_3(xR_3)^r],$$

If we now make use of the summation of a geometric series, then

$$(14) \qquad G = \frac{C_1}{1-xR_1} + \frac{C_2}{1-xR_2} + \frac{C_3}{1-xR_3}$$

$$= \frac{C_1(1-xR_2)(1-xR_3) + C_2(1-xR_1)(1-xR_3) + C_3(1-xR_1)(1-xR_2)}{1 - x(R_1+R_2+R_3) + x^2(R_1R_2 + R_1R_3 + R_2R_3) - x^3R_1R_2R_3},$$

which, upon employing the relations (9), (10), (11) and (12), finally reduces to the simple equation

$$(15) \qquad G = \frac{1-x}{1-2x-x^2+x^3} .$$

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