NUMERATOR POLYNOMIAL COEFFICIENT ARRAYS FOR CATALAN AND RELATED SEQUENCE CONVOLUTION TRIANGLES

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In this paper, we discuss numerator polynomial coefficient arrays for the row generating functions of the convolution arrays of the Catalan sequence and of the related sequences S_i [1], [2]. In three different ways we can show that those rows are arithmetic progressions of order i. We now unfold an amazing panorama of Pascal, Catalan, and higher arrays again interrelated with the Pascal array.

1. THE CATALAN CONVOLUTION ARRAY

The Catalan convolution array, written in rectangular form, is

Convolution Array for S_1

1	1	1	1	1	1	1	1	1	···
1	2	3	4	5	6	7	8	9	···
2	5	9	14	20	27	35	44	54	···
5	14	28	48	75	110	154	208	273	···
14	42	90	165	275	429	637	910	1260	···
42	132	297	572	1001	1638	···	···	···	···
···	···	···	···	···	···	···	···	···	···

Let $G_n(x)$ be the generating function for the n^{th} row, $n = 0, 1, 2, \cdots$. By the law of formation of the array, where C_{n-1} is a Catalan number,

$$G_{n-1}(x) = xG_n(x) - x^2G_n(x) + C_{n-1}\ .$$

Since

$$G_0(x) = 1/(1-x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

$$G_1(x) = 1/(1-x)^2 = 1 + 2x + 3x^3 + \cdots + (n+1)x^n + \cdots$$

we see that by the law of formation that the denominators for $G_n(x)$ continue to be powers of $(1-x)$. Thus, the general form is

$$G_n(x) = N_n(x)/(1-x)^{n+1}\ .$$

We compute the first few numerators as

$$N_1(x) = 1,\quad N_2(x) = 1,\quad N_3(x) = 2-x,\quad N_4(x) = 5-6x+2x^2,$$

$$N_5(x) = 14 - 28x + 20x^2 - 5,\quad \cdots$$

and record our results by writing the triangle of coefficients for these polynomials:

Numerator Polynomial $N_n(x)$ Coefficients Related to S_1

1							
1							
2	−1						
5	−6	2					
14	−28	20	−5				
42	−120	135	−70	14			
132	−495	770	−616	252	−42		
429	−2002	4004	−4368	2730	−924	132	
···	···	···	···	···	···	···	···

Notice that the Catalan numbers, or the sequence S_1, appears in the first column, and again as the bordering falling diagonal of the array. The next falling diagonal parallel to the Catalan numbers is the central diagonal of Pascal's triangle, taken with alternating signs and deleting the first one, or, the diagonal whose elements are given by $\binom{2n}{n}$. The rising diagonals, taken with the signs given, have sums 1, 1, 2, 4, 8, 16, 32, ⋯, 2^{n-1}, ⋯ . The row sums are all one. The coefficients for each row also can be used as a convolution with successive terms in rows of Pascal's triangle to write the terms in the rows for the convolution triangle. For example, the third row has coefficients 5, −6, and 2. The third row of Pascal's rectangular array is 1, 4, 10, 20, 35, 56, 84, ⋯, and we can obtain the third row of the convolution array for S_1 thus,

$$\begin{aligned}
5 &= 5\cdot 1 - 6\cdot 0 + 2\cdot 0\\
14 &= 5\cdot 4 - 6\cdot 1 + 2\cdot 0\\
28 &= 5\cdot 10 - 6\cdot 4 + 2\cdot 1\\
48 &= 5\cdot 20 - 6\cdot 10 + 2\cdot 4\\
75 &= 5\cdot 35 - 6\cdot 20 + 2\cdot 10\\
\cdots &\quad \cdots
\end{aligned}$$

We can take columns in the array of numerator polynomial coefficients to obtain columns in the Catalan convolution array. The zeroth or left-most column is already the Catalan sequence S_1. We look at successive columns:

$$\begin{aligned}
n &= 0 \qquad 1(1/1,\ 2/1,\ 5/1,\ 14/1,\ 42/1,\ \cdots) = 1,\ 2,\ 5,\ 14,\ 42,\ \cdots = S_1^2\\
n &= 1 \qquad 2(1/2,\ 6/3,\ 28/4,\ 120/5,\ 495/6,\ \cdots) = 1,\ 4,\ 48,\ 165,\ 572,\ \cdots = S_1^4\\
n &= 2 \qquad 3(2/6,\ 20/10,\ 135/15,\ 770/21,\ \cdots) = 1,\ 6,\ 27,\ 110,\ \cdots = S_1^6\\
n &= 3 \qquad 4(5/20,\ 70/35,\ 616/56,\ 4368/84,\ \cdots) = 1,\ 8,\ 44,\ 208,\ \cdots = S_1^8
\end{aligned}$$

The divisors are consecutive elements from column 1, column 2, and column 3 of Pascal's triangle. The first case could have divisors from the zeroth column of Pascal's triangle and is S_1^2. Thus, the i^{th} column of the numerator coefficient triangle for the S_1 array, the i^{th} column of the Pascal array, and the i^{th} column of the convolution array for S_1 are closely interrelated.

2. THE CONVOLUTION ARRAY FOR S_2

Next we write the numerator polynomial coefficient array for the generating functions for the rows of the convolution array for the sequence S_2. First, the convolution array for S_2 is

Convolution Array for S_2

1	1	1	1	1	1	1	1	1	⋯
1	2	3	4	5	6	7	8	9	⋯
3	7	12	18	25	33	42	52	62	⋯
12	30	55	88	130	182	245	320	408	⋯
55	143	273	455	700	1020	1428	1938	2565	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

The numerator polynomial coefficient array is

Numerator Polynomial Coefficients Related to S_2

1					
1					
3	−2				
12	−18	7			
55	−132	108	−30		
273	−910	1155	−660	143	
1428	−6120	⋯	⋯	⋯	⋯

Again, the row sums are one. The rising diagonals, taken with signs, have sums which are half of the sums of the rising diagonals, taken without signs, of the numerator polynomial coefficient array related to S_1. Again, the zeroth column is S_2, and the falling diagonal bordering the array at the top is S_2^2. The next falling diagonal is three times the diagonal 1, 6, 36, 220, ⋯, which is found in Pascal's triangle by starting in the third row of Pascal's triangle and counting right one and down two. (The diagonal in the corresponding position in the array related to S_1 is twice the diagonal 1, 3, 10, 35, 126, ⋯, which is found by starting in the first row and counting down one and right one in Pascal's rectangular array.)

Again, columns of the convolution array for S_2 arise from the columns of the numerator polynomial coefficient array, as follows:

$$n = 0 \quad 1(1/1, 3/1, 12/1, 55/1, \cdots) = 1, 3, 12, 55, \cdots = S_2^3$$

$$n = 1 \quad 2(2/4, 18/6, 132/8, 910/10, 6120/12, \cdots) = 1, 6, 33, 182, \cdots = S_2^6$$

$$n = 2 \quad 3(7/21, 108/36, 1155/55, \cdots) = 1, 9, 63, \cdots = S_2^9$$

$$n = 3 \quad 4(30/120, 660/220, 9282/364, \cdots) = 1, 12, 102, \cdots = S_2^{12}.$$

Note that the zeroth column could also be expressed as S_2^3, and could be obtained by multiplying the column by one and dividing successively by 1, 1, 1, ⋯. Each column above is divided by alternate entries of column 1, column 2, column 3 of Pascal's triangle. $S_2^{3(n+1)}$ is obtained by multiplying the n^{th} column of the numerator polynomial coefficient array by n and by dividing by every second term of the $(n-1)^{st}$ column of Pascal's triangle, $n = 0, 1, 2, \cdots$. Also notice that when the elements in the i^{th} row of the numerator array are convolved with i successive elements of the i^{th} row of Pascal's triangle written in rectangular form, we can write the i^{th} row of the convolution triangle for S_2.

3. The Convolution Array for S_3

For the next higher sequence S_3, the convolution array is

Convolution Array for S_3

1	1	1	1	1	1	1	1	1	⋯
1	2	3	4	5	6	7	8	9	⋯
4	9	15	22	30	39	49	60	72	⋯
22	52	91	140	200	272	357	456	570	⋯
140	340	612	969	1425	1995	2695	3542	4554	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

and the array of coefficients for the numerator polynomials for the generating functions for the rows is

Numerator Polynomial Coefficients Related to S_3

1					
1					
4	−3				
22	−36	15			
140	−360	312	−91	⋯	
⋯	⋯	⋯	⋯	⋯	⋯

Again, the first column is S_3, or, S_3^4, while the falling diagonal bordering the array is S_3^3, and the falling diagonal adjacent to that is four times the diagonal found in Pascal's triangle by beginning in the fifth row and counting right one and down three throughout the array, or, 1, 9, 78, 560, ⋯. The rising diagonal sums taken with signs, s_i, are related to the rising diagonal sums taken without signs, r_i, of the numerator array related to S_2 by the curious formula $r_i = 4s_i - i$, $i = 1, 2, \cdots$. Again, a convolution of the numerator coefficients in the i^{th} row with i elements taken from the i^{th} row of Pascal's triangle produces the i^{th} row of the convolution triangle for S_3. For example, for $i = 3$, we obtain the third row of the convolution array for S_3 as

$$\begin{aligned}
22 &= 22\cdot 1 - 36\cdot 0 + 15\cdot 0\\
52 &= 22\cdot 4 - 36\cdot 1 + 15\cdot 0\\
91 &= 22\cdot 10 - 36\cdot 4 + 15\cdot 1\\
140 &= 22\cdot 20 - 36\cdot 10 + 15\cdot 4\\
\cdots & \qquad \cdots
\end{aligned}$$

We obtain columns of the convolution array for S_3 from columns of the numerator polynomial coefficient array as follows:

$$\begin{aligned}
n &= 0 \quad 1(1/1, 4/1, 22/1, 140/1, \cdots) = 1, 4, 22, 140, \cdots = S_3^4\\
n &= 1 \quad 2(3/6, 36/9, 360/12, \cdots) = 1, 8, 60, \cdots = S_3^8\\
n &= 2 \quad 3(15/45, 312/78, 1560/120, \cdots) = 1, 12, 114, \cdots = S_3^{12}
\end{aligned}$$

Here, the divisors are every third element taken from column 0, column 1, column 2, ⋯ of Pascal's triangle.

4. THE GENERAL RESULTS FOR THE SEQUENCES S_i

These results continue. Thus, for S_i, the n^{th} column of the array of coefficients for the numerator polynomials for the generating functions of the rows of the S_i convolution array is multiplied by $(n+1)$ and divided by every i^{th} successive element in the n^{th} row of Pascal's rectangular array, beginning with the $[(n+1)i-1]^{st}$ term, to obtain the successive elements in the $(in+i-1)^{st}$ column of the convolution array for S_i, or the sequence $S_i^{i(n+1)}$. That is, we obtain the columns i, $2i+1$, $3i+2$, $4i+3$, ⋯, of the convolution array for S_i.

We write expressions for each element in each array in what follows, using the form of the m^{th} element of S_i^k given in [1].

Actually, one can be much more explicit here. The actual divisiors in the division process are

$$\binom{i(m+n)+(n-1)}{n},$$

where we are working with the sequence S_i, $i = 0, 1, 2, \cdots$; the n^{th} column of Pascal's triangle, $n = 0, 1, 2, \cdots$; and the m^{th} term in the sequence of divisors, $m = 1, 2, 3, \cdots$.

Now, we can write the elements of the numerator polynomial coefficient array for the row generating function of the convolution array for the sequence S_i. First, we write

$$S_i^k = \left\{\frac{k}{mi+i}\binom{(i+1)m+k-1}{m}\right\}, \quad m = 0, 1, 2, \cdots$$

which gives successive terms of the $(k-1)^{st}$ convolution of the sequence S_i. Then, when $k = (i+1)(n+1)$,

$$S_i^{(i+1)(n+1)} = \left\{\frac{(i+1)(n+1)}{mi+(i+1)(n+1)}\binom{(i+1)(m+n)+i}{m}\right\},$$

$$m = 0, 1, 2, \cdots; \qquad i = 0, 1, 2, \cdots; \quad n = 0, 1, 2, \cdots.$$

Let $a_{n+m,n}$ be the element in the numerator polynomial triangle for S_i, $m = 0, 1, 2, \cdots$, $n = 0, 1, 2, \cdots$, in the n^{th} column and $(n+m)^{th}$ row. Then, the topmost element in the n^{th} column is given by $a_{n,n}$. Now,

$$S_i^{(i+1)(n+1)} = \left\{(n+1)a_{n+m,n} \Big/ \binom{i(n+m)+n+i-1}{n}\right\}$$

so that, upon solving for $a_{n+m,n}$ after equating the two expressions for the m^{th} term of $S_i^{(i+1)(n+1)}$, we obtain

$$\begin{aligned}
a_{m+n,n} &= \frac{i+1}{i(m+n)+n+i+1}\binom{(i+1)(m+n)+i}{m}\binom{i(n+m)+n+i-1}{n}\\
&= \frac{i+1}{m}\binom{(i+1)(m+n)+i}{m-1}\binom{(i+1)n+(i-1)+mi}{n}.
\end{aligned}$$

Now, we can go from the convolution array to the numerator polynomial array, and from Pascal's triangle to the convolution array, and from Pascal's triangle directly to the numerator polynomial array.

And, do not fail to notice the beautiful sequences which arise from the first terms used for divisors in each column division for the columns of the numerator polynomial coefficients of this section. For the Catalan sequence S_1, the first divisors of successive columns were 1, 2, 6, 20, 70, ⋯, the central column of Pascal's triangle which gave rise to the Catalan numbers originally. For S_2, they are 1, 4, 21, 120, ⋯, which diagonal of Pascal's triangle yields S_2 upon successive division by $(3j+1)$, $j = 0, 1, 2, \cdots$, and $S_2^2 = \{1, 2, 7, 60, \cdots\}$ upon successive division by 1, 2, 3, 4, ⋯. For S_3, the first divisors are 1, 6, 45, ⋯, which produce $S_3^3 = \{1, 3, 15, 91, \cdots\}$, upon successive division by 1, 2, 3, 4, ⋯.

REFERENCES

1. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Catalan and Related Sequences Arising from Inverses of Pascal's Triangle Matrices," *The Fibonacci Quarterly,* Vol. 14, No. 5, pp. 395–404.
2. V. E. Hoggatt, Jr., and Marjorie Bicknell, "Pascal, Catalan, and General Sequence Convolution Arrays in a Matrix," *The Fibonacci Quarterly,* Vol. 14, No. 2, pp. 136–142.

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ON THE N CANONICAL FIBONACCI REPRESENTATIONS OF ORDER N

$$x^N - \sum_{i=0}^{N-1} x^i$$

for some $N \geq 2$. Then

$$a^{N+i} = \sum_{k=0}^{N-1} F_{N,i}^k a^{N-k}, \qquad i = 1, 2, 3, \cdots.$$

Proof. The case $i = 1$ amounts to $F_{N,1}^k = 1$, $k = 0, 1, \cdots, N-1$. If the theorem is true for some $i \geq 1$, then

$$a^{N+i+1} = \sum_{k=0}^{N-1} F_{N,i}^k a^{N-k+1} = \sum_{k=0}^{N-2} F_{N,i}^{k+1} a^{N-k} + F_{N,i}^0 a^{N+1} = \sum_{k=0}^{N-2} (F_{N,i}^{k+1} + F_{N,i}^0) a^{N-k} + F_{N,i}^0.$$

Now

$$F_{N,i}^{k+1} + F_{N,i}^0 = F_{N,i+k+1} - \sum_{j=0}^{k} F_{N,i+j} + F_{N,i} = F_{N,i+1+k} - \sum_{j=0}^{k-1} F_{N,i+1+j} = F_{N,i+1}^k.$$

Also $F_{N,i} = F_{N,i+1}^{N-1}$, so the above equation reduces to

$$a^{N+i+1} = \sum_{k=0}^{N-1} F_{N,i+1}^k a^{N-k}.$$

REFERENCES

1. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations of Higher Order," *The Fibonacci Quarterly,* Vol. 10 (1972), pp. 43–70.
2. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations of Higher Order II," *The Fibonacci Quarterly,* Vol. 10 (1972), pp. 71–80.
3. L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., "Fibonacci Representations," *The Fibonacci Quarterly,* Vol. 10 (1972), pp. 1–28.

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