FIBONACCI NUMBERS IN THE COUNT OF SPANNING TREES

PETER J. SLATER*
Applied Mathematics Division, National Bureau of Standards, Washington, D.C. 20234**

Hilton [3] and Fielder [1] have presented formulas for the number of spanning trees of a labelled wheel or fan in terms of Fibonacci and Lucas numbers. Each of them has also counted the number of spanning trees in one of these graphs which contain a specified edge. The purpose of this note is to generalize some of their results. The graph theory terminology used will be consistent with that in [2], F_k denotes the k^{th} Fibonacci number, and L_k denotes the k^{th} Lucas number. All graphs will be connected, and $ST(G)$ will denote the number of spanning trees of labelled graph, or multigraph, G.

A *fan* on k vertices, denoted N_k, is the graph obtained from path $P_{k-1} = 2, 3, \cdots, k$ by making vertex 1 adjacent to every vertex of P_{k-1}. The *wheel* on k vertices, denoted W_k, is obtained by adding edge $(2,k)$ to N_k. That is, $W_k = N_k + (2,k)$. A *planar* graph G is one that can be drawn in the plane so that no two edges intersect; G is *outerplanar* if it can be drawn in the plane so that no two edges intersect, and all its vertices lie on the same face; and a *maximal outerplanar graph* G is an outerplanar graph for which $G + (u,v)$ is not outerplanar for any pair u,v of vertices of G such that edge (u,v) is not already in G. For example, each fan is a maximal outerplanar graph because, as will be used in the proof of Proposition 1, an outerplanar graph on k vertices is maximal outerplanar if and only if it has $2k - 3$ edges.

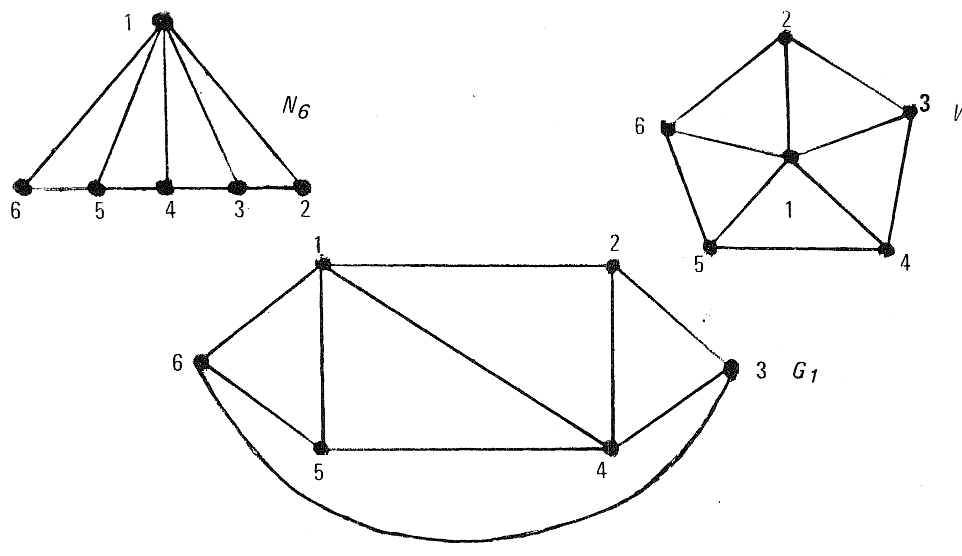

Figure 1 Three Graphs on Six Vertices

As shown in Hilton [3], $ST(N_k) = F_{2k-2}$ and $ST(W_k) = L_{2k-2} - 2$. Let OP_k^j denote the set of maximal outerplanar graphs with k vertices, of which exactly j are of degree two. Note that $N_k \in OP_k^2$ for $k \geq 4$, and, with G_1 as in Figure 1, $G_1 - (3,6) \in OP_6^2$.

*This work was done while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.
**Author is currently at Sandia Laboratories, Applied Mathematics Division-5121, Albuquerque, N.M. 87115.

Proposition 1. If $H \in OP_k^2$, then $ST(H) = F_{2k-2}$.

Proof. If k equals 4 or 5, then $OP_k^2 = \{N_k\}$, and $ST(N_k) = F_{2k-2}$ for any k. The proposition will be proved by induction on k. Suppose it is true for $4 \leqslant k \leqslant n-1$ with $n \geqslant 6$, and suppose $H \in OP_n^2$. Assume the vertices of H are labelled so that $1, 2, \cdots, n$ is a cycle bounding the outside face and vertex n is one of the two vertices of degree two, written $\deg(n) = 2$. Now H is maximally outerplanar implies that edge $(1, n-1)$ is in H. Also, either $(1, n-2)$ or $(n-1, 2)$ is in H, and, by symmetry, one can assume $(1, n-2)$ is in H. (See Fig. 2.)

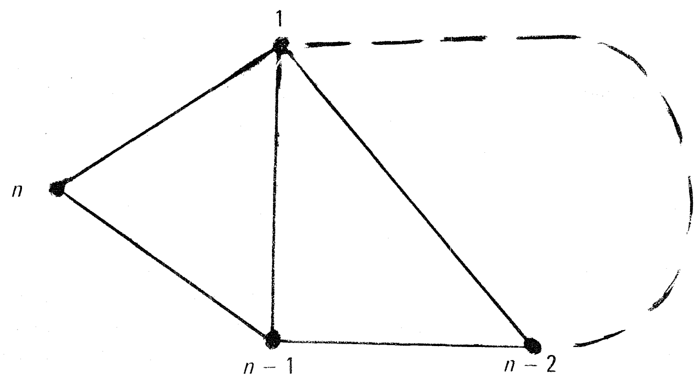

Figure 2 Structure in a Graph $H \in OP_n^2$

Since any spanning tree T contains at least one edge incident with vertex n, either T is a spanning tree of $H-(1,n)$ or $H-(n-1, n)$, or else T contains both edges $(1,n)$ and $(n-1, n)$. Now $\deg(n-1) = 2$ in $H-n$ implies $H - n \in OP_{n-1}^2$. Hence, $ST(H-(1,n)) = ST(H-(n-1,n)) = ST(H-n) = F_{2n-4}$. Also, $\deg(1) \geqslant 3$ and $\deg(n-2) \geqslant 3$ in H, but exactly one of these two vertices will have degree two in $H - \{n, n-1\}$, that is, $H - \{n, n-1\} \in OP_{n-2}^2$. Now the number of spanning trees of H using both $(1,n)$ and $(n-1, n)$ equals the number of spanning trees of $H-n$ using $(1, n-1)$. This is obtained by subtracting the number of spanning trees of $H-n$ that contain $(n-1, n-2)$ but not $(1, n-1)$ from the total number of spanning trees of $H-n$, and one obtains

$$F_{2n-4} - ST(H-n-(1,n-1)) = F_{2n-4} - ST(H-\{n,n-1\}) = F_{2n-4} - F_{2n-6} = F_{2n-5}.$$

Consequently,

$$ST(H) = ST(H-(1,n)) + ST(H-(n-1,n)) + F_{2n-5} = 2F_{2n-4} + F_{2n-5} = F_{2n-2},$$

and the proposition is proved.

For OP_k^j with $j \geqslant 3$, no result like Proposition 1 is possible. Indeed, let $H_1 = N_7 + 8 + (8,4) + (8,5)$, and let $H_2 = N_7 + 8 + (8,3) + (8,4)$. Then $H_1 \in OP_8^3$, $H_2 \in OP_8^3$, $ST(H_1) = 368$ and $ST(H_2) = 369$.

Allowing there to be several edges connecting each pair of vertices, let G be any multigraph. Several observations can be helpful.

Observation 1. Suppose v is a cutpoint of (connected) multigraph G, and $G - v$ has components $C_1, C_2, \cdots, C_t$. If B_i is the subgraph of G induced by C_i and v $(1 \leqslant i \leqslant t)$, then

$$ST(G) = \prod_{i=1}^{t} ST(B_i).$$

For example, vertex 1 is a cutpoint of $N_6 - (3,4)$, and $ST(N_6 - (3,4)) = ST(N_3) \cdot ST(N_4) = 3 \cdot 8 = 24$.

Observation 2. Suppose (u,v) is an edge of multigraph G and G' is obtained from G by identifying u and v and deleting (u,v). (Note that even if G is a graph then G' may have multiple edges. Also, if (u,v) is one of several edges between u and v, then G' will have loops, but no spanning tree contains a loop.) Then $ST(G')$ is the number of spanning trees of G that contain edge (u,v). For example, $ST(W_{k+j+1})$ is the number of spanning trees of "biwheel" $W_{k,j}$ (as in Fig. 3) which contain the edge (u,v).

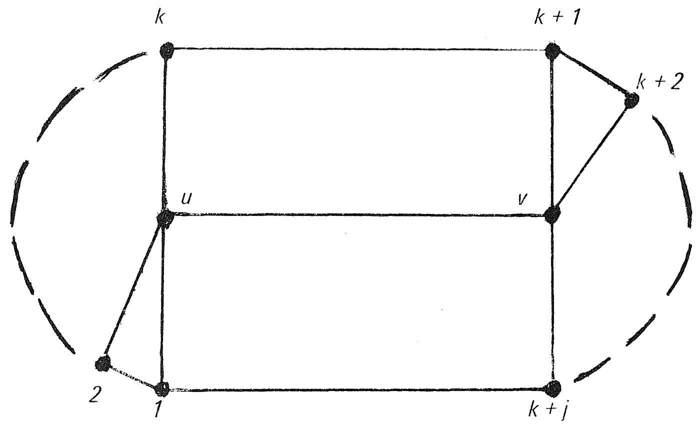

Figure 3 A "Biwheel" on $k + j + 2$ Vertices with $1 \leqslant j \leqslant k$ and $k \geqslant 2$

Observation 3. Suppose edge (u,v) is in spanning tree T of G. Let U (respectively, V) by the subgraph of G induced by the set of vertices in the component of $T - (u,v)$ that contains u (respectively, v). Clearly there are $ST(U)\cdot ST(V)$ labelled spanning trees of G containing (u,v) that produce these same two subgraphs. This presents another way to count the labelled spanning trees of G containing (u,v). For example, in graph G_1 of Fig. 1, let $u = 3$ and $v = 6$. The possibilities for the vertex set of U are

$$\{3\}, \{3,4\}, \{3,4,5\}, \{3,2\}, \{3,2,4\}, \{3,2,4,5\}, \{3,2,1\}, \{3,2,4,1\} \text{ and } \{3,2,4,1,5\}.$$

Thus one obtains

$$1\cdot 21 + 1\cdot 3 + 1\cdot 1 + 1\cdot 8 + 3\cdot 3 + 3\cdot 1 + 1\cdot 1 + 8\cdot 1 + 21\cdot 1 = 75$$

spanning trees containing (3,6).

Let G be any multigraph, and let G' be as in Observation 2, then $ST(G) = ST(G - (u,v)) + ST(G')$. That is, $ST(G)$ is given by evaluating the number of spanning trees in two multigraphs, each one with fewer edges and one with one fewer vertices. As this procedure can be iterated, one can compute $ST(G)$ in this manner for any multigraph G.

One can also find formulas for classes of graphs, such as the "biwheels," where the biwheel on $k + j + 2$ vertices, denoted $W_{k,j}$, is as in Fig. 3 with deg $(u) = k + 1$ and deg $(v) = j + 1$.

Let U (respectively, V) be the fan N_k (respectively, N_j) containing u (respectively, v) in

$$H = W_{k,j} - \{(k, k+1), (u,v), (1, k+j)\}.$$

Consider the spanning trees of $W_{k,j}$ that contain $(k, k+1)$ and $(1, k+j)$ but not (u,v). Any such spanning tree of $W_{k,j}$ contains a spanning tree of U or V, but not both. The number of such spanning trees that contain a fixed spanning tree of V can be found, using a slight variation of Observation 3, by enumerating the number of spanning subgraphs of U that have two components, each of which is a tree, one containing vertex 1 and the other containing vertex k. This equals $2(ST(N_k) + ST(N_{k-1}) + \cdots + ST(N_2))$. Similarly, if $j \geqslant 2$, there are

$$2(ST(N_j) + ST(N_{j-1}) + \cdots + ST(N_2))$$

such spanning trees containing a fixed spanning tree of U.

Proposition 2. $ST(W_{k,j}) = L_{2k+2j} + 2F_{2k+2j} - 2F_{2j} - 2F_{2k} - 2.$

Proof. The number of spanning trees of $W_{k,j}$ which contain (u,v) is $ST(W_{k+j+1})$. The number of spanning trees containing $(k, k+1)$ but not (u,v) or $(1, k+j)$ (or $(1, k+j)$ but not $(k, k+1)$ or (u,v)) is

$$ST(N_{k+1})\cdot ST(N_{j+1}).$$

Thus

$$ST(W_{k,1}) = L_{2k+2} - 2 + 2F_{2k} + 2(F_2 + F_4 + \cdots + F_{2k-2}),$$

and, if $j \geq 2$,

$$ST(W_{k,j}) = L_{2k+2j} - 2 + 2F_{2k}F_{2j} + 2F_{2j}(F_2 + F_4 + \cdots + F_{2k-2}) + 2F_{2k}(F_2 + F_4 + \cdots + F_{2j-2}).$$

Simple Fibonacci identities reduce these equations to the desired formula.

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THE DIOPHANTINE EQUATION $(x_1 + x_2 + \cdots + x_n)^2 = x_1^3 + x_2^3 + \cdots + x_n^3$

W. R. UTZ
University of Missouri – Columbia, Mo.

The Diophantine equation

$$(x_1 + x_2 + \cdots + x_n)^2 = x_1^3 + x_2^3 + \cdots + x_n^3 \tag{1}$$

has the non-trivial solution $x_i = i$ as well as permutations of this n-tuple since

$$\sum_{i=1}^{n} i = n(n+1)/2 \quad \text{and} \quad \sum_{i=1}^{n} i^3 = n^2(n+1)^2/4.$$

Also, for any n, $x_i = n$ for all $i = 1, 2, \cdots, n$, is a solution of (1). Thus, (1) has an infinite number of non-trivial solutions in positive integers.

On the other hand if one assumes $x_i > 0$, then for each i one has $x_i < n^2$. To see this, let a be the largest coordinate in a solution $(x_1, x_2, \cdots, x_n)$. Then,

$$x_1 + x_2 + \cdots + x_n \leq na.$$

For the same solution

$$x_1^3 + x_2^3 + \cdots + x_n^3 \geq a^3$$

and so $a \leq n^2$. Thus, we see that for a fixed positive integer, n, equation (1) has only a finite number of solutions in positive integers and we have proved the following theorem.

Theorem. Equation (1) has only a finite number of solutions in positive integers for a fixed positive integer n but as $n \to \infty$ the number of solutions is unbounded.

Clearly if $(x_1, x_2, \cdots, x_n)$ is a solution of (1) wherein some entry is zero, then one has knowledge of a solution (1) for $n - 1$ and so, except for $n = 1$, we exclude all solutions with a zero coordinate hereafter.

[Continued on page 16.]