ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-358 Proposed by Phil Mana, Albuquerque, New Mexico.

Prove that the integer u_n such that $u_n \leqslant n^2/3 < u_n + 1$ is a prime for only a finite number of positive integers n. (Note that $u_n = [n^2/3]$, where $[x]$ is the greatest integer in x and that $u_1 = 0$, $u_2 = 1$, $u_3 = 3$, $u_4 = 5$, and $u_5 = 8$.)

B-359 Proposed by R. S. Field, Santa Monica, California.

Find the first three terms T_1, T_2, and T_3 of a Tribonacci sequence of positive integers $\{T_n\}$ for which

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} (T_n/10^n) = 1/T_4 .$$

B-360 Proposed by T. O'Callahan, Aerojet Manufacturing Co., Fullerton, California.

Show that for all integers a, b, c, d, e, f, g, h there exist integers w, x, y, z such that

$$(a^2 + 2b^2 + 3c^2 + 6d^2)(e^2 + 2f^2 + 3g^2 + 6h^2) = (w^2 + 2x^2 + 3y^2 + 6z^2).$$

B-361 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{r,s=0}^{\infty} x^r y^s u^{\min(r,s)} v^{\max(r,s)}$$

is a rational function of x, y, u, and v when these four variables are less than 1 in absolute value.

B-362 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let m be an integer greater than one and let R_n be the remainder when the triangular number $T_n = n(n+1)/2$ is divided by m. Show that the sequence $R_0, R_1, R_2, \cdots$ repeats in a block $R_0, R_1, \cdots, R_t$ which reads the same from right to left as it does from left to right. (For example, if $m = 7$ the smallest repeating block is 0, 1, 3, 6, 3, 1, 0.)

B-363 Proposed by Herta T. Freitag, Roanoke, Virginia.

Do the sequences of squares $S_n = n^2$ and of pentagonal numbers $P_n = n(3n-1)/2$ also have the symmetry property stated in B-362 for their residues modulo m?

SOLUTIONS

THE PRIMES PETER OUT

B-334 Proposed by Phil Mana, Albuquerque, New Mexico.

Are all the terms prime in the sequence 11, 17, 29, 53, ⋯ defined by $u_0 = 11$, $u_{n+1} = 2u_n - 5$ for $n > 0$?

Composite of solutions by David G. Beverage, San Diego Evening College, La Mesa, California and Heiko Harborth, Technische Universität Braunschweig, West Germany.

One easily sees that $u_8 = 1541 = 23 \cdot 67$ is composite. More interestingly, one can show by induction that $u_n = 5 + 6 \cdot 2^n$. Then $u_n = 17 + 6(2^{n-1} - 1)$ and $2^4 \equiv -1 \pmod{17}$ and so $17 | u_{8k+1}$ for $k = 1, 2, \cdots$. Also, the Fermat Theorem tells us that $2^{p-1} \equiv 1 \pmod{p}$ for odd primes p and this can be used to show divisibility properties such as $11 | u_{10k}$ and $19 | u_{18k+11}$.

Also solved by George Berzsenyi, Wray G. Brady, Paul S. Bruckman, Dinh Thế Hung, Sidney Kravitz, H. Turner Laquer, D. P. Laurie, Graham Lord, John W. Milsom, T. Ponnudurai, Bob Prielipp, Jeffrey Shallit, Sahib Singh, Paul Smith, Gregory Wulczyn, David Zeitlin, and the Proposer.

FIBONACCI-LUCAS SUM

B-335 Proposed by Herta T. Freitag, Roanoke, Virginia.

Obtain a closed form for

$$\sum_{i=0}^{n-k} (F_{i+k}L_i + F_iL_{i+k}).$$

Solution by Graham Lord, Université Laval, Québec, Canada.

The sum multiplied by $\sqrt{5}$ equals

$$\sum_{i=0}^{n-k} [(a^{i+k} - b^{i+k})(a^i + b^i) + (a^i - b^i)(a^{i+k} + b^{i+k})] = 2\sum_{i=0}^{n-k} (a^{2i+k} - b^{2i+k})$$

$$= 2[a^{2n-k+1} - a^{k-1} - (b^{2n-k+1} - b^{k-1})].$$

Hence the closed form is $2(F_{2n-k+1} - F_{k-1})$.

Also solved by David G. Beverage, Wray G. Brady, Paul S. Bruckman, Ralph Garfield, Dinh Thế Hung, H. Turner Laquer, A. G. Shannon, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

PELL SQUARES

B-336 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let $Q_0 = 1 = Q_1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$. Show that $2(Q_{2n}^2 - 1)$ is a perfect square for $n = 1, 2, 3, \cdots$.

Solution by H. Turner Laquer, University of New Mexico, Albuquerque, New Mexico.

By induction $2(Q_{2n}^2 - 1) = (Q_{2n} + Q_{2n-1})^2$ for $n = 1, 2, \cdots$ giving $2(Q_{2n}^2 - 1)$ as a perfect square.

Also solved by George Berzsenyi, David G. Beverage, Wray G. Brady, Paul S. Bruckman, Ralph Garfield, Dinh Thế Hung, Sidney Kravitz, Graham Lord, T. Ponnudurai, Bob Prielipp, Jeffrey Shallit, A. G. Shannon, Gregory Wulczyn, David Zeitlin, and the Proposer.

RATIONAL POINTS ON AN ELLIPSE

B-337 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Show that there are infinitely many points with both x and y rational on the ellipse $25x^2 + 16y^2 = 82$.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

We shall establish the stronger result that if a rational number $r \neq 0$ is the sum of the squares of two rational numbers, then it has infinitely many representations as the sum of the squares of two positive rational numbers.

First, let $r = a^2 + b^2$, where a and b are rational numbers both different from zero. Without loss of generality, we may assume that a and b are both positive and that $a \geqslant b$. For every positive integer k,

$$(*) \qquad r = \left(\frac{(k^2-1)a-2kb}{k^2+1}\right)^2 + \left(\frac{(k^2-1)b+2ka}{k^2+1}\right)^2 .$$

If $k \geqslant 3$, $3k^2 - 8k = 3k(k-3) + k \geqslant 3$ and hence $3(k^2-1) \geqslant 8k$. Thus

$$\frac{k^2-1}{2k} \geqslant \frac{4}{3} > 1 \geqslant \frac{b}{a}$$

so $(k^2-1)a > 2kb$, from which it follows immediately that

$$a_k = \frac{(k^2-1)a-2kb}{k^2+1} > 0.$$

If $j > k$, where j and k are positive integers then

$$(j^2-k^2)a + b(kj-1)(j-k) > 0.$$

But this is equivalent to

$$\frac{(j^2-1)a-2jb}{j^2+1} > \frac{(k^2-1)a-2kb}{k^2+1} .$$

Therefore the numbers

$$a_k = \frac{(k^2-1)a-2kb}{k^2+1}$$

increase with k so the a_k's are all different. Hence when $k \geqslant 3$ (*) gives different representations of r as the sum of the squares of two positive rational numbers.

Also solved by David G. Beverage, Paul S. Bruckman, H. Turner Laquer, Bob Prielipp, Sahib Singh, Paul Smith, Gregory Wulczyn, and the Proposer.

DIFFERENCE OF BINOMIAL EXPANSIONS

B-338 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Let k and n be positive integers. Let $p = 4k + 1$ and let h be the largest integer with $2h + 1 \leqslant n$. Show that

$$\sum_{j=0}^{h} p^j \binom{n}{2j+1}$$

is an integral multiple of 2^{n-1}.

Solution by H. Turner Laquer, University of New Mexico, Albuquerque, New Mexico.

Let

$$M(n,k) = \sum_{j=0}^{h} p^j \binom{n}{2j+1} .$$

As

$$(1+x)^n = \sum_{j=0}^{n} x^j \binom{n}{j} \quad \text{and} \quad (1-x)^n = \sum_{j=0}^{n} (-1)^j x^j \binom{n}{j}$$

one has

$$M(n,k) = ((1+\sqrt{p})^n - (1-\sqrt{p})^n)/(2\sqrt{p}).$$

Using this and the fact that $(1 \pm \sqrt{p})^2 = 2 \pm 2\sqrt{p} + 4k$, one obtains

$$M(n,k)/2^{n-1} = M(n-1,k)/2^{n-2} + kM(n-2,k)/2^{n-3}.$$

As $M(1,k) = 1$ and $M(2,k) = 2$ one can use induction to prove that $M(n,k)$ is divisible by 2^{n-1}.

Also solved by David G. Beverage, Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, David Zeitlin, and the Proposer.

OPERATIONAL IDENTITY

B-339 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Establish the validity of E. Cesàro's symbolic Fibonacci-Lucas identity $(2u+1)^n = u^{3n}$; after the binomial expansion has been performed, the powers of u are used as either Fibonacci or Lucas subscripts. (For example, when $n = 2$ one has both $4F_2 + 4F_1 + F_0 = F_6$ and $4L_2 + 4L_1 + L_0 = L_6$.)

Solution by Graham Lord, Université Laval, Québec, Canada.

For a fixed K, since both

$$F_K a + F_{K-1} = a^k \quad \text{and} \quad F_K b + F_{K-1} = b^K,$$

the n^{th} power of each when added (algebraically) will give the result

$$(F_K u + F_{K-1})^n = u^{Kn}.$$

The desired equation is the special case when $K = 3$.

Also solved by David G. Beverage, Wray G. Brady, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, H. Turner Laquer, A. G. Shannon, David Zeitlin, and the Proposer.

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[Continued from page 284.]

Solution by David Beverage, San Diego Community College, San Diego, California.

By using the polynomials $P_{2n+1}(x)^*$ expressed explicitly as

$$(1) \qquad P_{2n+1}(x) = \sum_{r=0}^{n} 5^{n-r}(-1)^{kr} \frac{(2n+1)[(2n-r)!]}{r!\,(2n+1-2r)!} x^{2n+1-2r} \; {**}$$

and selecting $m = 2n+1$, obtain

$$(2) \qquad Q = \frac{F_{mp}}{F_p} = F_p \cdot H \pm m,$$

where H is a polynomial in F_p.

Clearly,

$$(F_p,m) \mid (F_p,Q).$$

Select $m > 1$ with integral coefficients and $m \mid F_p$ $(m \not\equiv 0\ (p))$ in order that $(F_p,Q) > 1 \cdots$. The above conditions are satisfied for many numbers m and p. One example: $p = 7$ and $m = 13$ produces

$$\frac{F_{91}}{F_7} = 358465123875040793 = Q \quad \text{and} \quad (F_7, Q) = 13 > 1.$$

Many other interesting divisor relationships may be obtained from the polynomials $P_{2n+1}(x)$.

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*David G. Beverage, "A Polynomial Representation of Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 9 No. 5 (Dec. 1971)

**David G. Beverage, "Polynomials $P_{2n+1}(x)$ Satisfying $P_{2n+1}(F_k) = F_{(2n+1)k}$," *The Fibonacci Quarterly,* Vol. 14, No. 3 (Oct. 1976), pp. 197–200.

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