COMPOSITIONS AND RECURRENCE RELATIONS II

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In an earlier paper by the same authors [1] properties of the compositions of an integer with 1 and 2 were discussed. This paper is a sequel to the earlier one and contains results on modes and related concepts. We stress once again as before that the word "compositions" refers only to compositions with ones and twos unless specially mentioned.

Definition 1. To every composition of a positive integer N we add an unending string of zeroes at both ends. The transition $\cdots 0+1+\cdots$ is a rise while $\cdots + 1 + 0 + \cdots$ is a fall. We also defined in [1] that a one followed by a two is rise while it is a fall if they occur in reverse order. We also define $\cdots 0 + 1 + \cdots + 1 + 2$ as *a* rise and $\cdots 2 + 1 + \cdots + 1 + 0 + \cdots$ as *a* fall.

Definition 2. A composition of a positive integer N is called "unimaximal" if there is exactly one rise and one fall. In other words it is unimaximal if there is no 1 occurring between two 2's. (All the 2's are bunched together.) Let $M^1(N)$ denote the number of unimaximal (unimax in short) compositions of N.

Definition 3. A composition of a positive integer is called "uniminimal" if there is no 2 occurring between two 1's. (All the 1's are bunched together.) Let $m^1(N)$ denote the number of uniminimal (unimin in short) compositions of N.

We shall now investigate some of the properties of $m^1(N)$ and $M^1(N)$ and make an asymptotic estimate of $m^1(N)/M^1(N)$.

Theorem 1.

(a) $$M^1(N) = M^1(N-1) + [N/2]$$

(b) $$m^1(N) = m^1(N-2) + [N/2]$$

(c) $$M^1(2N) = \frac{M^1(2N+1) + M^1(2N-1)}{2}$$

(d) $$m^1(2N) + m^1(2N-1) = m^1(2N+1) + m^1(2N-2),$$

where $[x]$ represents the largest integer $\leqslant x$.

Proof. Let $M^1(N,1)$ and $M^1(N,2)$ denote the number of unimax compositions ending with 1 and 2, respectively. Clearly $M^1(N) = M^1(N,1) + M^1(N,2)$. By Definition 2 we see that

(1) $$M^1(N,1) = M^1(N-1)$$

since the 1 at the end of the compositions counted by $M^1(N,1)$ will not affect the bunching of twos. However a 2 at the end preserves unimax if and only if it is preceded by another 2 or a complete string of ones only. Thus

(2) $$M^1(N,2) = M^1(N-2,2) + 1$$

so that decomposing (2) further we arrive at

$$M^1(2N+1) = N$$

and

(3) $$M^1(2N) = N.$$

Putting (1) and (3) together we get

(4) $$M^1(N) = M^1(N-1) + [N/2].$$

Now using similar combinatorial arguments for m^1 with similar notation for $m^1(N,1)$ and $m^1(N,2)$ we see

(5) $$m^1(N) = m^1(N,1) + m^1(N,2)$$

and

(6) $$m^1(N,2) = m^1(N-2)$$

while

$$m^1(N,1) = m^1(N-1,1) + 1 \quad \text{if} \quad N-1 \equiv 0 \pmod 2$$
$$m^1(N,1) = m^1(N-1,1) \quad \text{if} \quad N \equiv 1 \pmod 2$$

which gives

(7) $$m^1(2N) = m^1(2N-2) + N$$

(8) $$m^1(2n+1) = m^1(2N-1) + N$$

or

$$m^1(N) = m^1(N-2) + [n/2].$$

From (4) we deduce

$$M^1(2N) = \frac{M^1(2N+1) + M^1(2N-1)}{2}$$

for

$$M^1(2N) = M^1(2N-1) + N$$
$$M^1(2N+1) = M^1(2N) + N.$$

Finally (7) and (8) together imply

$$m^1(2N) + m^1(2N-1) = m^2(2N+1) + m^1(2N-2)$$

proving Theorem 1.

Theorem 2.

$$\lim_{N\to\infty} \frac{m^1(N)}{M^1(N)} = \frac{1}{2}.$$

Proof. Let Δ_n denote the n^{th} triangular number

$$\Delta_n = \frac{n(n+1)}{2}.$$

In general for real x let

(9) $$\Delta_x = \frac{x(x+1)}{2}.$$

It is not difficult to establish using induction and Theorem 1 that

(10) $$m^1(2N+1) = \Delta_{N+1}$$

(11) $$m^1(2N) = m^1(2N-1) + 1$$

so that (10) and (11) together imply

(12) $$m^1(N) = \Delta_{N/2} + O(1).$$

One can also show similarly that

(13) $$M^1(2N+1) = \Delta_{N+1} + \Delta_{N-1}$$

and

(14) $$M^1(2N) = \frac{M^1(2N+1) + M^1(2N-1)}{2} = \frac{\Delta_{n+1} + 2\Delta_{N-1} + \Delta_{N-3}}{2}$$

which give

(15) $$M^1(N) = 2\Delta_{N/2} + O(N)$$

for

$$\lim_{N\to\infty} \Delta_N/\Delta_{N+1} = 1.$$

Now (12) and (15) together imply

$$\lim_{N\to\infty} \frac{m^1(N)}{M^1(N)} = \frac{1}{2}$$

proving Theorem 2.

Definition 4. Every rise and a fall determines a maximum. Every fall and a rise determines a minimum. Let $M(N)$ and $m(N)$ denote the number of maximums and minimums in the compositions of N.

Theorem 3.

$$M(N) = M(N-1)+M(N-2)+F_{N-2}-1$$
$$m(N) = m(N-1)+m(N-2)+F_{N-2}-1$$
$$\lim_{N\to\infty} \frac{m(N)}{M(N)} = 1.$$

Proof. As before split $M(N)$ as

$$M(N) = M(N,1)+M(N,2).$$

It is clear that the "1" at the end of the compositions counted by $M(N,1)$ does not record a max and so

$$M(N,1) = M(N-1).$$

Clearly the "2" at the end of the compositions counted by $M(N,2)$ records an extra max if and only if the corresponding composition counted by $N-2$ ends in a 1 but not for $N-2 = 1+1+\cdots 1$ a string of ones. Thus

$$M(N,2) = M(N-2)+C_{N-2}(1)-1$$
$$= M(N-2)+F_{n-2}-1$$

giving

(16) $$M(N) = M(N-1)+M(N-2)+F_{N-2}-1.$$

Proceeding similarly for $m(N)$ we have

$$m(N) = m(N,1)+m(N,2) \quad \text{and} \quad m(N,1) = m(N-1)+C_{N-1}(2)-1 = m(N-1)+F_{N-2}-1$$

while $m(N,2) = m(N-2)$ giving

(17) $$m(N) = m(N-1)+m(N-2)+F_{N-2}-1.$$

It is quite clear from (16) and (17) that $m(N)$ and $M(N)$ are Fibonacci Convolutions so that [see Hoggatt and Alladi [2]].

(18) $$\lim_{N\to\infty} \frac{F_N}{m(N)} = \lim_{N\to\infty} \frac{F_N}{M(N)} = 0.$$

Now pick any composition of N say N_C. Let $M(N_C)$ and $m(N_C)$ denote the number of max and min, respectively in N_C. Since there is a fall between two rises and a rise between two falls we have

(19) $$|M(N_C)-m(N_C)| \leq 1.$$

Now from the definition of N_C it is obvious that

(20) $$|M(N)-m(N)| = \left|\sum_C M(N_C) - \sum_C m(N_C)\right| = \left|\sum_C (M(N_C)-m(N_C))\right| \leq \sum_C |M(N_C)-m(N_C)|$$
$$\leq C_N = F_{N+1}$$

by (19). Now if we use (18) we get

$$\lim_{N\to\infty} \frac{m(N)}{M(N)} = 1.$$

In other words the number of maximums and the number of minimums are asymptotically equal.

Let us now find the asymptotic distribution of 1's and 2's in unimax compositions. Let $M_1(N)$ and $M_2(N)$ denote the number of ones and number of twos in the unimax compositions of N.

Theorem 4.

$$M_1(2N+1) = M_1(2N)+M^1(2N)+N^2, \quad M_1(2N) = M_1(2N-1)+M^1(2N-1)+N(N-1).$$

Proof. As before, let

(21) $$M_1(N) = M_1(N,1) + M_1(N,2).$$

Clearly we have

$$M_1(N,1) = M_1(N-1) + M^1(N-1)$$

while

(22) $$M_1(N,2) = M_1(N-2,2) + (N-2)$$

for the compositions $1+1+1 \cdots 1 = N-2$, and $1+1+\cdots 1+2 = N$ are both unimax. Now if we decompose (22) further we sum alternate integers. Then (21) gives the two equations of Theorem 4.

Theorem 5. $$M_2(2N+1) = M_2(2N) + N + \frac{(N-1)N}{2}$$

$$M_2(2N) = M_2(2N-1) + N + \frac{(N-1)N}{2}.$$

Proof. By combinatorial arguments similar to Theorem 4 we get

$$M_2(N) = M_2(N,1) + M_2(N,2)$$

giving $M_2(N,1) = M_2(N-1)$ and

$$M_2(N,2) = M_2(N-2,2) + M^1(N-2,2) + 1 = N/2 + M^1(N-2,2) + M^1(N-4,2) + \cdots$$

on further decomposition. We also know from (3) that

$$M^1(2N+1,2) = M^1(2N,2) = N$$

so that

$$M_2(2N+1) = M_2(2N) + \frac{N(N+1)}{2}, \quad M_2(2N) = M_2(2N-1) + \frac{N(N+1)}{2}.$$

Theorem 6.

$$\lim_{N\to\infty} \frac{M_2(N)}{M_1(N)} = \frac{1}{2}.$$

Proof. It is easy to prove that for real x

(23) $$f(x) = \sum_{N \leqslant x} N^2 \sim \frac{x^3}{3}$$

We know from Theorem (4) that

(24) $$M_1(2N+1) = M_1(2N) + M^1(2N) + N^2$$

(25) $$M_1(2N) = M_1(2N-1) + M^1(2N-1) + N(N-1).$$

From (4) one can deduce without trouble that

(26) $$M^1(2N+1) = N^2 + N + 1$$

(27) $$M^1(2N) = N^2 + 1.$$

Now substituting (26) and (27) in (24) and (25) and continuing the decomposition using the recursion on Theorem 4 we get

(28) $$M_1(N) = \sum_{m \leqslant N/2} m^2 + \sum_{m \leqslant N/2} m^2 + O(N^2) = \frac{2}{3}\left(\frac{N}{2}\right)^3 + O(N^2) \sim \frac{2}{3}\left(\frac{N}{2}\right)^3$$

using (23). If we adopt the same decomposition procedure to the two equations in Theorem (5) we get by virtue of

(29) $$M_2(N) = \sum_{m \leqslant N/2} m^2 + O(N^2) = \frac{1}{3}\left(\frac{N}{2}\right)^3 + O(N^2).$$

Now (28) and (29) together imply

$$\lim_{N\to\infty} \frac{M_2(N)}{M_1(N)} = \frac{1}{2}$$

establishing Theorem 6,

We now state theorems analogous to (4) and (5) and (6) for the uniminimal compositions.

Theorem 7.

$$m_1(N) = m_1(N-2) + \sum_{n=1}^{N-1} m^1(n,1) + [N/2]$$

$$m_2(N) = m_2(N-2) + m^1(N-2) + [N/2]\,.$$

Proof. With the usual notation $m_1(N,1)$ and $m_1(N,2)$ we find

$$m_1(N) = m_1(N,1) + m_1(N,2)$$

$$m_1(N_1 2) = m_1(N-2)$$

since the "2" at the end of the compositions counted by $m(N,2)$ will not affect the counting of minimums or ones. However for $m_1(N,1)$ we find

$$\begin{aligned} m_1(N,1) &= m_1(N-1,\,1) + m^1(N-1,\,1) + 1 \\ &\quad \text{if } N-1 \equiv 0 \pmod 2 \\ &= m_1(N-1,\,1) + m^1(N-1,\,1) \\ &\quad \text{if } N-1 \equiv 1 \pmod 2 \end{aligned}$$

so putting these together we get

$$m_1(N) = m_1(N-2) + \sum_{n=1}^{N-1} m^1(n,1) + [N/2]\;.$$

With similar use of notation for m_2 we get

$$m_2(N) = m_2(N,1) + m_2(N,2)$$

giving

$$m_2(N_1 2) = m_2(N-2) + m^1(N-2)$$

while

$$\begin{aligned} m_2(N,1) &= m_2(N-1,\,1) + 1 \quad \text{if } N-1 \equiv 0 \pmod 2 \\ &= m_2(N-1) \qquad\qquad \text{if } N-1 \equiv 1 \pmod 2 \end{aligned}$$

so that these give

$$m_2(N) = m_2(N-2) + m^1(N-2) + [N/2]\,.$$

Theorem 8.

$$\lim_{N\to\infty} \frac{m_2(N)}{m_1(N)} = \frac{1}{2}\,.$$

Proof. We know from Theorem 7 that

(29)
$$m_1(N) = m_1(N-2) + \sum_{n=1}^{N-1} m^1(n,1) + [n/2]\,.$$

Now from Theorem 1 we deduce that

$$m^1(n,1) = [n/2]$$

so that

(30)
$$\sum_{n=1}^{N-1} m^1(n,1) = \sum_{n=1}^{N-1} [n/2] = \sum_{n=1}^{N-1} [n/2] + O(N-1) = \frac{\Delta_{N-1}}{2} + O(N-1)\,.$$

If we continue to decompose $m_1(N-2)$ in (29) and use (30) we will finally get

(31) $$m_1(N) = \left\{\frac{\Delta_{N-1}}{2} + \frac{\Delta_{N-3}}{2} + \frac{\Delta_{N-5}}{2} + \dots\right\} + O(N^2) \sim \frac{\Delta_{N-1}}{2} + \frac{\Delta_{N-3}}{2} + \dots .$$

We also know from Theorem 7 that

(32) $$m_2(N) = m_2(N-2) + m^1(N-2) + [N/2]$$

It is easy to establish from Theorem 1 that

$$m^1(2N+1) = \Delta_{N+1}, \qquad m^1(2N+2) = m^1(2N+1) + 1$$

giving

(33) $$m^1(N) = \Delta_{N/2} + O(N) \sim \Delta_{N/2} .$$

Now decomposing $m_2(N-2)$ in (32) further and using (33) we get

(34) $$m_2(N) = \left\{\frac{\Delta_{N-2}}{2} + \frac{\Delta_{N-4}}{2} + \dots\right\} + O(N^2) = \frac{1}{2}\left\{\frac{\Delta_{N-2}}{2} + \frac{\Delta_{N-4}}{2} + \dots\right\} + O(N^2)$$

$$= \frac{1}{2}\left\{\frac{\Delta_{N-1}}{2} + \frac{\Delta_{N-3}}{2} + \dots\right\} + O(N^2)$$

since $x \sim y$ implies $\Delta_x \sim \Delta_y$. Now if we compare (34) and (31) we get

$$\lim_{N\to\infty} \frac{m_2(N)}{m_1(N)} = \frac{1}{2}$$

proving Theorem 8.

We now shift our attention to compositions called "Zeckendorf compositions." A composition of N in which no two consecutive ones appear is called a Zeckendorf composition (1) and if no two consecutive twos appear it is called a Zeckendorf composition (2). We denote them in short as z_1 and z_2 compositions respectively. Note that in a z_1 composition there *should be a* 2 between ones while in a unimin there *should not* similarly z_2 is the opposite of unimax. Now denote by

$Z(N)$ = the number of Z_2 compositions of N

$z(N)$ = the number of Z_1 compositions of N.

Theorem 9. $Z(N) = Z(N-1) + Z(N-3), \qquad z(N) = z(N-2) + z(N-3).$

$$\lim_{N\to\infty} \frac{z(N)}{Z(N)} = 0.$$

Proof. As usual partition

$$z(N) = z(N,1) = z(N,2)$$

clearly $z(N,2) = z(N-2)$, while

$$z(N,1) = z(N-1,2) = z(N-3)$$

this proves

$$z(N) = z(N-2) + z(N-3).$$

Again

$$Z(N) = Z(N_1 1) + Z(N,2) \quad \text{and} \quad Z(N,1) = Z(N-1)$$

while

$$Z(N,2) = Z(N-2,1) = Z(N-3)$$

giving

$$Z(N) = Z(N-1) + Z(N-3).$$

It can be shown that

$$\lim_{N\to\infty} \frac{Z(N+1)}{Z(N)} = a$$

and

$$\lim_{N\to\infty} \frac{Z(N+1)}{z(N)} = \beta,$$

where a and β are the dominant roots of the auxiliary polynomials $x^3 - x^2 - 1 = 0$ and $x^3 - x - 1 = 0$ $(a > \beta)$. See Hoggatt and Alladi [2]. This implies that there exist constants $c_\alpha, c_\beta > 0$ so that

$$Z(N) > C_\alpha a^N$$

and

and

$$z(N) < C_\beta \beta^N$$

giving

$$\lim_{N \to \infty} \frac{z(N)}{Z(N)} = 0.$$

Corollary. On similar lines

$$\lim_{N \to \infty} \frac{Z(N)}{C_N} = \lim_{N \to \infty} \frac{z(N)}{C_N} = 0.$$

NOTE. Given a partition of N in terms of 1 and 2, if we rearrange the summands so as to get the maximum number of max we get a Z_2 composition. If we rearrange to get the maximum number of min we get a Z_1 composition. Roughly a Zeckendorf composition is either a maximax or a maximin composition.

REFERENCES

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A TOPOLOGICAL PROOF OF A WELL KNOWN FACT ABOUT FIBONACCI NUMBERS

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Theorem. Let p be a prime. Then there is a sequence $\{m_j\}$ of positive integers such that

$$F_{m_j} \equiv 1 - F_{m_j - 1} \equiv 1 - F_{m_j + 1} \equiv 0 \pmod{p^j}.$$

The proof depends on the following lemma.

Lemma. Let G be a topological group whose completion (in the natural uniformity) is compact. Let $g \in G$. Then the sequence $g, g^2, g^3, \cdots$ has a subsequence which converges to 1.

Proof. The sequence of powers of g has an accumulation point $h = \lim_{j \to \infty} g^{n_j}$ in the compact completion $\bar{G}$ of G. Let $m_j = n_{j+1} - n_j$. Then $g^{m_j} \to 1$ in $\bar{G}$ and hence in G.

To prove the theorem we shall apply the lemma to

$$g = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

in the group G of 2×2 integer matrices of determinant ± 1 topologized p-adically. That is, for every integer n write $n = p^k m$, $(p,m) = 1$ and set $\|n\|_p = p^{-k}$. Then for $A, B \in G$ let

$$d(A,B) = \max \left\{ \|A_{ij} - B_{ij}\|_p : i, j = 1, 2 \right\}$$

G equipped with the metric d satisfies the hypotheses of the lemma.

It is easy to check inductively that

$$g^m = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix}.$$

[Continued on p. 280.]