POLYNOMIALS ASSOCIATED WITH CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

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BACKGROUND

Jaiswal [1] investigated certain polynomials $p_n(x)$ related to Chebyshev polynomials of the second kind $U_n(x)$ for which

(1) $$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad U_0(x) = 1, \quad U_1(x) = 2x$$

with

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \; .$$

In this article, similar properties are derived for the corresponding polynomials $q_n(x)$ related to Chebyshev polynomials of the first kind $T_n(x)$ for which

(2) $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0(x) = 2, \quad T_1(x) = 2x$$

with

$$T_n(\cos\theta) = 2\cos n\theta \; .$$

The first few Chebyshev polynomials of the first kind are

(3) $$\begin{cases} T_0(x) = 2 \\ T_1(x) = 2x \\ T_2(x) = 4x^2 - 2 \\ T_3(x) = 8x^3 - 6x \\ T_4(x) = 16x^4 - 16x^2 + 2 \\ T_5(x) = 32x^5 - 40x^3 + 10x \\ T_6(x) = 64x^6 - 96x^4 + 36x^2 - 2 \\ T_7(x) = 128x^7 - 224x^5 + 112x^3 - 14x \\ T_8(x) = 256x^8 - 512x^6 + 320x^4 - 64x^2 + 2 \\ \text{------------------------------} \end{cases}$$

THE ASSOCIATED POLYNOMIALS

Now take the sums along the rising diagonals on the right-hand side of (3). We obtain polynomials $q_n(x)$ which bear a close relationship to the Fibonacci numbers F_n. It is natural to define $q_0(x) = 0$.

From (3), the first few polynomials $q_n(x)$ are

(4) $$\begin{cases} q_1(x) = 2, \quad q_2(x) = 2x, \quad q_3(x) = 4x^2, \quad q_4(x) = 8x^3 - 2, \quad q_5(x) = 16x^4 - 6x, \\ q_6(x) = 32x^5 - 16x^2, \quad q_7(x) = 64x^6 - 40x^3 + 2, \quad q_8(x) = 128x^7 - 96x^4 + 10x, \\ q_9(x) = 256x^8 - 224x^5 + 36x^2, \quad q_{10}(x) = 512x^9 - 512x^6 + 112x^3 - 2 \; . \end{cases}$$

Observe in (4) the recurrence relation

(5) $$q_{n+3}(x) = 2xq_{n+2}(x) - q_n(x) \quad (n \geqslant 0)$$

which is (not unexpectedly) similar to Jaiswal's recurrence relation.

SOME PROPERTIES OF THE POLYNOMIALS

The $q_n(x)$ are seen to be connected with Jaiswal's $p_n(x)$ by the formula

(6) $$q_n(x) = p_n(x) - p_{n-3}(x) \quad (n \geqslant 3,\ p_0(x) = 0)$$

leading to

(7) $$\sum_{n=3}^{\infty} q_n(x)t^n = \sum_{n=3}^{\infty} p_n(x)t^n - \sum_{n=3}^{\infty} p_{n-3}(x)t^n \quad (n \geqslant 3)$$

i.e., by Jaiswal's generating function, to the generating function

(8) $$\sum_{n=3}^{\infty} q_n(x)t^n = (t - t^4)(1 - 2xt + t^3)^{-1}$$

For convenience, write the left-hand side of (8) as

(9) $$Q(x,t) = \sum_{n=3}^{\infty} q_n(x)t^n$$

from which we have (abbreviating $Q(x,t)$ as Q)

(10) $$\frac{\partial Q}{\partial t} = \frac{1 - 6t^3 - t^6 + 6xt^4}{(1 - 2xt + t^3)^2}, \qquad \frac{\partial Q}{\partial x} = \frac{t - t^4}{(1 - 2xt + t^3)^2}$$

Manipulation with (10) leads to the partial differential equation

(11) $$2t\,\frac{\partial Q}{\partial t} - (2x - 3t^2)\,\frac{\partial Q}{\partial x} - 8Q + 6G_1 = 0,$$

where, adjusting Jaiswal's notation slightly, we write

$$G_1(x,t) = \sum_{n=3}^{\infty} p_n(x)t^n = \frac{t}{1 - 2xt + t^3} .$$

But from (9),

(12) $$\frac{\partial Q}{\partial t} = \sum_{n=3}^{\infty} nq_n(x)t^{n-1}, \qquad \frac{\partial Q}{\partial x} = \sum_{n=3}^{\infty} q'_n(x)t^n .$$

Substitution in (11) yields

(13) $$2xq'_{n+2}(x) - 3q'_n(x) = 2(n-2)q_{n+2}(x) + 6p_{n+2}(x) \quad (n \geqslant 0).$$

Comparing coefficients of t^{n+1} in (8), we obtain

$$q_{n+1}(x) = (2x)^n - \binom{n-2}{1}(2x)^{n-3} + \binom{n-4}{2}(2x)^{n-6} - \cdots - \left\{(2x)^{n-3} - \binom{n-5}{1}(2x)^{n-6} + \cdots\right\}$$

that is,

(14) $$q_{n+1}(x) = \sum_{r=0}^{\left[\frac{n}{3}\right]} \binom{n-2r}{r}(-1)^r(2x)^{n-3r} \sum_{r=0}^{\left[\frac{n-3}{3}\right]} \binom{n-3-2r}{r}(-1)^r(2x)^{n-3-3r} .$$

SPECIAL CASE $x = 1$

Putting $x = 1$ in (4) and writing $Q_n \equiv q_n(1)$, we obtain the sequence

(15)

	n=0	1	2	3	4	5	6	7	8	9	10	···
Q_n:	0	2	2	4	6	10	16	26	42	68	110	···
	≡2(0	1	1	2	3	5	8	13	21	34	55	···)

Clearly

(16) $$Q_n = 2F_n \quad (n \geqslant 0),$$

where F_n is the n^{th} Fibonacci number.

It might be remarked that when $x = 1$, Eq. (5) becomes

$$Q_{n+3} = 2Q_{n+2} - Q_n \quad (n \geqslant 0)$$

which is a characteristic feature of the Fibonacci sequence of numbers.

[Setting $x = 1$ in $\{U_n\}$ and $\{T_n\}$ gives, on using (1) and (2) (or (3)), the sequences 1, 2, 3, 4, 5, 6, ⋯ and 2, 2, 2, 2, 2, 2, ⋯, respectively.]

Further, one may notice that

$$P_n = Q_n + F_{n-1} - 1, \tag{17}$$

where P_n are the numbers obtained from Jaiswal's polynomials $p_n(x)$ by putting $x = 1$, i.e., $P_n \equiv p_n(1)$.

$$(P_{n+1} = P_{n+1} + P_n - 1, \qquad P_0 = 1, \qquad P_1 = 1.)$$

Finally, $x = 1$ in (14) yields, with (16),

$$F_{n+1} = \tfrac{1}{2}\left\{\sum_{r=0}^{[n/3]} \binom{n-2r}{r}(-1)^r 2^{n-3r} - \sum_{r=0}^{\left[\frac{n-3}{3}\right]} \binom{n-3-2r}{r}(-1)^r 2^{n-3-3r}\right\} . \tag{18}$$

Our results should be compared with the corresponding results produced by Jaiswal. The generating function (8), and the properties which flow from it such as (11) and (13), are slightly less simple than we might have wished. However, the Fibonacci property (16) could hardly be simpler. What we lose on the swings we gain on the roundabouts!

REFERENCE

1. D. V. Jaiswal, "On Polynomials Related to Tchebichef Polynomials of the Second Kind," *The Fibonacci Quarterly,* Vol. 12, No. 3 (Oct. 1974), pp. 263–265.

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Proposed by Guy A. R. Guillot, Montreal, Quebec, Canada.

Show that

$$\text{(a)} \qquad \frac{\pi}{2} = \sum_{n=1}^{\infty} \tan^{-1} \frac{2F_{2n+1}}{F_{2n}F_{2n+2}}$$

$$\text{(b)} \qquad \frac{\pi}{2} = \sum_{n=1}^{\infty} \cos^{-1} \frac{F_{2n}F_{2n+2}}{F_{2n}F_{2n+2}+2}$$

$$\text{(c)} \qquad \frac{\pi}{2} = \sum_{n=1}^{\infty} \sin^{-1} \frac{2F_{2n+1}}{F_{2n}F_{2n+2}+2} .$$

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Find a function A_k in terms of k alone for the following expression.

$$F_n = \sum_{k=1}^{F_n} p_k - \sum_{k=1}^{F_n} A_k ,$$

where p_k denotes the k^{th} prime and F_n denotes the n^{th} Fibonacci number.

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