ZERO-ONE SEQUENCES AND STIRLING NUMBERS OF THE SECOND KIND

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Let $x_1, x_2, \cdots, x_n$ denote a sequence of zeros and ones of length n. Define a polynomial of degree $(n-m) \geqslant 0$ as follows

(1) $$\beta_{m+1,n+1}(d) = \sum d^{1-x_1}(d+x_1)^{1-x_2} \cdots (d+x_1+x_2+\cdots+x_{n-1})^{1-x_n}$$

with $\beta_{1,1}(d) = 1$, where the summation is over $x_1, x_2, \cdots, x_n$ such that

$$\sum_{i=1}^{n} x_i = m.$$

Summing over x_n we have the following recurrence relation

(2) $$\beta_{m+1,n+1}(d) = (m+d)\beta_{m+1,n}(d) + \beta_{m,n}(d),$$

where $\beta_{0,0}(d) = 1$.

Summing over x_1 we have the following recurrence relation

(3) $$\beta_{m+1,n+1}(d) = d\cdot\beta_{m+1,n}(d) + \beta_{m,n}(d+1),$$

where $\beta_{0,0}(d) = 1$.

Now we introduce the following theorems to establish relationships between the polynomials defined in (1) and Stirling numbers of the second kind; see Riordan [1, pp. 32–34].

Theorem 1. $\beta_{m,n}(1)$ defined in (1) is Stirling numbers of the second kind, i.e., $\beta_{m,n}(1)$ is the coefficient of $t^n/n!$ in the expansion of $(e^t - 1)^m/m!$, $m,n \geqslant 1$.

Proof. From (1) we have $\beta_{1,1}(1) = 1$ and from (2) we have

(4) $$\beta_{m+1,n+1}(1) = (m+1)\beta_{m+1,n}(1) + \beta_{m,n}(1),$$

which is the recurrence relation for Stirling numbers of the second kind; see Riordan [1, p. 33]. Thus Theorem 1 is proved.

Using (2), (3), and (4), we have

Corollary 1.

(a) $$\beta_{m+1,n+1}(0) = \beta_{m,n}(1),$$

(b) $$\beta_{m+1,n+1}(1) = \beta_{m+1,n}(1) + \beta_{m,n}(2),$$

(c) $$\beta_{m,n}(2) = m\beta_{m+1,n}(1) + \beta_{m,n}(1).$$

Theorem 2. The polynomial defined in (1) can be written

$$\beta_{m+1,n+1}(d) = \sum_{y=0}^{(n-m)} \binom{n}{y} d^y \beta_{m,n-y}(1).$$

Proof. Assume that n distinguishable balls are randomly distributed into N distinguishable cells such that the probability a ball falls in a specified cell is $1/N$. Assume that $d = \theta N \leqslant N$, $0 \leqslant \theta \leqslant 1$, of the cells are previously occupied.

Define $x_i = 1$ if i^{th} ball falls in an empty cell and $x_i = 0$ otherwise. The joint probability function of $(x_1, x_2, \cdots, x_n)$ can be written

(5)
$$\left(\frac{N-d}{N}\right)^{x_1}\left(\frac{d}{N}\right)^{1-x_1}\left(\frac{N-d-x_1}{N}\right)^{x_2}\left(\frac{d+x_1}{N}\right)^{1-x_2}\cdots$$
$$\cdots\left(\frac{N-d-x_1-x_2-\cdots-x_{n-1}}{N}\right)^{x_n}\left(\frac{d+x_1+x_2+\cdots+x_{n-1}}{N}\right)^{1-x_n}.$$

Let $E_{m,j,k}$ be the event that m additional cells will be occupied when j balls are randomly distributed into k cells such that the probability that a ball falls in a specified cell is $1/k$. Now summing (5) over $x_1, x_2, \cdots, x_n$ such that

$$\sum_{i=1}^{n} x_i = m,$$

we have

(6)
$$P[E_{m,n,N}] = \frac{1}{N^n}\frac{(N-d)!}{(N-d-m)!}\beta_{m+1,n+1}(d).$$

Let $F_{y,n}$ denote the event that y out of n balls will fall in the previously occupied cells, d out of N cells. Then

(7)
$$P[F_{y,n}] = \binom{n}{y}\left(\frac{d}{N}\right)^{y}\left(1-\frac{d}{N}\right)^{n-y}, \qquad y = 0, 1, \cdots, n.$$

But we have

$$P[E_{m,n,N}] = \sum_{y=0}^{(n-m)} P[F_{y,n}]P[E_{m,n,N}|F_{y,n}],$$

where using similar expression as (5) and (a) of Corollary 1,

(8)
$$P[E_{m,n,N}|F_{y,n}] = P[E_{m,n-y,N-d}] = \frac{1}{(N-d)^{n-y}}\frac{(N-d)!}{(N-d-m)!}\beta_{m,n-y}(1).$$

Thus using (7) and (8)

(9)
$$P[E_{m,n,N}] = \frac{1}{N^n}\frac{(N-d)!}{(N-d-m)!}\left\{\sum_{y=0}^{(n-m)}\binom{n}{y}d^y\beta_{m,n-y}(1)\right\}.$$

Equating (6) and (9), Theorem 2 follows.

From Theorem 2, we have the following recurrence relation for Stirling numbers of the second kind.

Corollary 2.

$$\beta_{m+1,n+1}(1) = \sum_{y=0}^{(n-m)}\binom{n}{y}\beta_{m,n-y}(1)$$

REFERENCE

1. John Riordan, *An Introduction to Combinatorial Analysis,* Wiley, New York, 1958.

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