ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-276 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that the sequence of Bell Numbers, $\{B_i\}_{i=0}^{\infty}$, is invariant under repeated differencing.

$$B_0 = 1, \qquad B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \qquad (n \geqslant 0).$$

H-277 Proposed by L. Taylor, Brentwood, New York.

If $p \equiv \pm 1 \pmod{10}$ is prime and $x \equiv \sqrt{5}$ is of even order (mod p), prove that $x-3, x-2, x-1, x, x+1$ and $x+2$ are quadratic nonresidues of p if and only if $p \equiv 39 \pmod{40}$.

SOLUTIONS

A PLAYER REP

H-261 Proposed by A. J. W. Hilton, University of Reading, Reading, England.

It is known that, given k a positive integer, each positive integer n has a unique representation in the form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t} ,$$

where $t = t(n,k)$, $a_i = a_i(n,k)$, $(i = t, \cdots, k)$, $t \geqslant 1$ and, if $k > t$, $a_k > a_{k-1} > \cdots > a_t$. Call such a representation the *k-binomial representation* of n.

Show that, if $k \geqslant 2$, $n = r+s$, where $r \geqslant 1$, $s \geqslant 1$ and if the k-binomial representations of r and s are

$$r = \binom{b_k}{k} + \binom{b_{k-1}}{k-1} + \cdots + \binom{b_u}{u}, \qquad s = \binom{c_k}{k} + \binom{c_{k-1}}{k-1} + \cdots + \binom{c_v}{v}$$

then

$$\binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1} \leqslant \binom{b_k}{k-1} + \binom{b_{k-1}}{k-2} + \cdots + \binom{b_u}{b-1} + \binom{c_k}{k-1} + \binom{c_{k-1}}{k-2} + \cdots + \binom{c_v}{v-1}.$$

Solution by the Proposer.

Define a total order $<_s$ on the collection of all k-sized sets of positive integers as follows: If A,B are two distinct sets of k positive integers write $A <_s B$ if

$$\max\{x : x \in A/B\} < \max\{x : x \in B/A\} .$$

Let $S_k(r)$ denote the collection of the first r sets under $<_s$, and let $S'_k(s)$ denote the collection of the first s sets under $<_s$ which do not contain any of $\{1, \cdots, r\}$. If $\mathcal{A}$ is any collection of n k-sized sets of positive integers let

$$\Delta\mathcal{A} = \{B : |B| = k-1 \quad \text{and} \quad B \subset A \ \text{ for some } A \in \mathcal{A}\} .$$

The Kruskal-Katona theorem states that $|\Delta A| \geqslant |\Delta S_k(n)|$. Thus

$$|\Delta S_k(n)| \leqslant |\Delta(S_k(r) \cup S'_k(s))|\ .$$

But

$$|\Delta S_k(n)| = \binom{a_k}{k-1} + \cdots + \binom{a_t}{t-1}$$

and

$$|\Delta(S_k(r) \cup S'_k(s))|$$

$$= |\Delta S_k(r)| + |\Delta S_k(s)| = \binom{b_k}{k-1} + \binom{b_{k-1}}{k-2} + \cdots + \binom{b_u}{u-1} + \binom{c_k}{k-1} + \binom{c_{k-1}}{k-2} + \cdots + \binom{c_v}{v-1}.$$

and the required inequality now follows.

REFERENCES

G.O.H. Katona, "A Theorem of Finite Sets, Theory of Graphs," *Proc. of Colloquium,* Tihany, Hungary (1966), pp. 187–207.

J. B. Kruskal, "The Number of Simplices in a Complex," *Mathematical Optimization Techniques,* University of California Press, Berkeley and Los Angeles (1963), pp. 251–278.

MODERN MOD

H-262 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that $L_{p^2} \equiv 1 \pmod{p^2}$ if and only if $L_p \equiv 1 \pmod{p^2}$.

Solution by the Proposer.

Put

$$L_n = \alpha^n + \beta^n, \qquad \alpha + \beta = 1, \qquad \alpha\beta = -1.$$

Then

$$1 = (\alpha+\beta)^n = L_n + \sum_{k=1}^{n-1} \binom{n}{k} \alpha^k \beta^{n-k}.$$

In particular

$$L_p = 1 - \sum_{k=1}^{p-1} \binom{p}{k} \alpha^k \beta^{p-k}$$

and

$$L_{p^2} = 1 - \sum_{k=1}^{p^2-1} \binom{p^2}{k} \alpha^k \beta^{p^2-k}.$$

Since

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} \quad \text{and} \quad \binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod{p},$$

it follows that

$$L_p \equiv 1 \pmod{p^2}$$

if and only if

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \alpha^k \beta^{p-k} \equiv 0 \pmod{p}.$$

In the next place

$$\binom{p^2}{k} \equiv 0 \pmod{p^2} \qquad (p \nmid k)$$

and

$$\binom{p^2}{pk} = \frac{p}{k}\binom{p^2-1}{pk-1} \equiv (-1)^{k-1}\frac{p}{k} \pmod{p^2}.$$

Thus $L_{p^2} \equiv 1 \pmod{p^2}$ if and only if

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\alpha^{pk}\beta^{p^2-pk} \equiv 0 \pmod{p}.$$

Since

$$\left(\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\alpha^{k}\beta^{p-k}\right)^p \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\alpha^{pk}\beta^{p^2-pk} \pmod{p},$$

it follows that $L_{p^2} \equiv 1 \pmod{p}$ if and only if

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\alpha^{k}\beta^{p-k} \equiv 0 \pmod{p}.$$

Therefore

$$L_{p^2} \equiv 1 \pmod{p^2} \Leftrightarrow L_p \equiv 1 \pmod{p^2}.$$

REMARK: More generally, if $k \geq 2$, we have

$$L_{p^k} \equiv 1 \pmod{p^2} \Leftrightarrow L_p \equiv 1 \pmod{p^2}.$$

LUCAS THE SQUARE IS NOW MOD!

H-263 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $L^2_{2mn} \equiv 4 \pmod{L^2_m}$ for every $n,m = 1, 2, 3, \cdots$.

Solution by the Proposer.

Clearly,

$$L_{(2k+1)m} = (\alpha^m)^{2k+1} + (\beta^m)^{2k+1}$$

is divisible by $L_m = \alpha^m + \beta^m$, where

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

Consequently, if n is odd, say $2k+1$, then upon using formulae (I_{15}) and (I_{18}) of Hoggatt's *Fibonacci and Lucas Numbers,*

$$L_{2nm} = L^2_{nm} - 2(-1)^{nm} = L^2_{(2k+1)m} - 2(-1)^m$$

and, depending upon the parity of m, either $L_{2nm}+2$ or $L_{2nm}-2$ is equal to $L^2_{(2k+1)m}$. Hence the product $(L_{2nm}+2)(L_{2nm}-2) = L^2_{2nm}-4$ is divisible by L^2_m.

If n is even, say $n = 2k$, we proceed by induction on k. For $k = 1$,

$$L_{2nm} = L_{4m} = L^2_{2m} - 2 = (L^2_m - 2(-1)^m)^2 - 2 = L^4_m - 4(-1)^m L^2_m + 2,$$

hence, $L_{2nm} - 2$ and, therefore, $L^2_{2nm} - 4$ is divisible by L^2_m. Assume now that the desired result holds for all even integers less than $n = 2k$. Then

$$L_{2nm} = L_{4km} = L^2_{2km} - 2,$$

and hence

$$L_{2nm} - 2 = L^2_{2km} - 4.$$

This latter expression is divisible by L^2_m either by the induction hypothesis or by the proof for odd n, thus $(L_{2nm}+2)(L_{2nm}-2)$ must also be divisible by L^2_m. This completes the inductive step.

Also solved by G. Lord, D. Beverage, F. Higgins, and G. Wulczyn.

AN OLDIE!

H-256 Proposed by E. Karst, Tucson, Arizona.

Find all solutions of

(i) $$x + y + z = 2^{2n+1} - 1,$$

and

(ii) $$x^3 + y^3 + z^3 = 2^{6n+1} - 1,$$

simultaneously for $n < 5$, given that

(a) x, y, z are positive rationals

(b) $2^{2n+1} - 1$, $2^{6n+1} - 1$ are integers

(c) $n = \log_2 \sqrt{t}$, where t is a positive integer.

Solution by the Proposer.

From this journal (Dec., 1972, p. 634; April, 1973, p. 188) we have the following

$$n, \qquad x + y + z = 2^{2n+1} - 1, \qquad x^3 + y^3 + z^3 = 2^{6n+1} - 1:$$

1. $1 + 1 + 5 = 7 = 2^3 - 1$ $\qquad 1^3 + 1^3 + 5^3 = 127 = 2^7 - 1$
2. $1 + 11 + 19 = 31 = 2^5 - 1$ $\qquad 1^3 + 11^3 + 19^3 = 8191 = 2^{13} - 1$
3. $1 + 55 + 71 = 127 = 2^7 - 1$ $\qquad 1^3 + 55^3 + 71^3 = 524287 = 2^{19} - 1$
4. $19 + 29 + 79 = 127 = 2^7 - 1$ $\qquad 19^3 + 29^3 + 79^3 = 524287 = 2^{19} - 1$
5. $1 + 239 + 271 = 511 = 2^9 - 1$ $\qquad 1^3 + 239^3 + 271^3 = 33554431 = 2^{25} - 1$.

Through the courtesy of Hans Riesel, Stockholm, we have also:

$$n = \log_2 \sqrt{34}, \qquad x,y,z = 13/2,\ 19,\ 83/2$$
$$n = \log_2 \sqrt{46}, \qquad x,y,z = 11,\ 47/2,\ 113/2$$
$$n = \log_2 \sqrt{76}, \qquad x,y,z = 26,\ 31,\ 94$$
$$n = \log_2 \sqrt{79}, \qquad x,y,z = 29,\ 121/4,\ 391/4 .$$

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