# FIXED POINTS OF CERTAIN ARITHMETIC FUNCTIONS

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## INTRODUCTION

Perfect, amicable and sociable numbers are fixed points of the arithemetic function $L$ and its iterates.

$$L(n) = \sigma(n) - n,$$

where $\sigma$ is the sum of divisor's function. Recently there have been investigations into functions differing from $L$ by 1; i.e., functions $L_+$, $L_-$, defined by $L_\pm(n) = L(n) \pm 1$. Jerrard and Temperley [1] studied the existence of fixed points of $L_+$ and $L_-$. Lal and Forbes [2] conducted a computer search for fixed points of $(L_-)^2$. For earlier references to $L_-$, see the bibliography in [2].

We consider the analogous situation using $\sigma^*$, the sum of unitary divisors function. Let $L_+^*$, $L_-^*$, be arithmetic functions defined by

$$L_\pm^*(n) = \sigma^*(n) - n \pm 1.$$

In § 1, we prove, using parity arguments, that $L_-^*$ has no fixed points.

Fixed points of iterates of $L_-^*$ arise in sets where the number of elements in the set is equal to the power of $L_-^*$ in question. In each such set there is at least one natural number $n$ such that $L_-^*(n) > n$. In § 2, we consider conditions $n$ must satisfy to enjoy the inequality and how the inequality acts under multiplication. In particular if $n$ is even, it is divisible by at least three primes; if odd, by five. If $n$ enjoys the inequality, any multiply by a relatively prime factor does so. There is a bound on the highest power of $n$ that satisfies the inequality. Further if $n$ does not enjoy the inequality, there are bounds on the prime powers multiplying $n$ which will yield the inequality.

In § 3, we describe a computer search for fixed points of iterates of $L_-^*$. In the range $0 < n < 110{,}000$, we found no sets of fixed points.

In § 4, we summarize theory and a computer search for $L_+^*$. Again, by a parity argument, we prove there are only two fixed points, 1, 2, for $L_+^*$. The computer search, $0 < n < 100{,}000$, found no fixed points of iterates of $L_+^*$.

## 1. THE FUNCTION $L_-^*$

Let $\underline{\underline{Z}}$ be the integers and $\underline{\underline{N}}$ the natural numbers. The arithmetic function $\sigma^*: \underline{\underline{N}} \to \underline{\underline{Z}}$ is the sum of unitary divisors function. For $n = \Pi p^\alpha$,

(1) $$\sigma^*(n) = \Pi(1 + p^\alpha).$$

Define new arithmetic functions $L^*$, $L_-^*$, $L_+^*$, by

(2) $$L^*(n) = \sigma^*(n) - n;$$

(3) $$L_-^*(n) = L^*(n) - 1;$$

(4) $$L_+^*(n) = L^*(n) + 1.$$

We are interested in the fixed points of $L_-^*$, $L_+^*$, and their iterates. For $L_-^*$, we call these fixed points *reduced unitary perfect* and *reduced unitary amicable* and *sociable numbers*. For $L_+^*$, *augmented unitary perfect, amicable* and *sociable numbers*. The names are suggested by [2]. We consider $L_-^*$ in detail.

Note that $L_-^*(1) = -1$ and $L_+^*(2) = 0$.

*Lemma 1.* For $n \in N$, $L_-^*(n) = 0$ if, and only if, $n = p^\alpha$, $p$ a prime, $p \geqslant 2$, $\alpha \geqslant 1$.

*Proof.* If $n = p^\alpha$, $L_-^*(n) = \sigma^*(n) - (1+n) = (1+n) - (1+n) = 0$.
If $n \neq p^\alpha$, $n = p^\alpha m$, $p$ a prime, $\alpha \geqslant 1$; $(p,m) = 1$, $m > 1$. Then $L_-^*(n) = \sigma^*(n) - (1+n)$.
But
$$\sigma^*(n) = \sigma^*(p^\alpha)\sigma^*(m)$$
$$\geqslant (1+p^\alpha)(1+m) = 1 + p^\alpha + m + p^\alpha m$$
$$L_-^*(n) \geqslant p^\alpha + m > 0.$$

*Lemma 2.* For $n \in N$, $L_-^*(n)$ has the same parity as $n$ if, and only if, $n = 2^\alpha$, $\alpha \geqslant 0$.

*Proof.* If $n = 2^0 = 1$; $L_-^*(1) = -1$.
If $n = 2^\alpha > 1$; $L_-^*(2^\alpha) = 0$ by Lemma 1.
If $n = \Pi p^\alpha$, all $p$ odd primes:
$L_-^*(n) = \Pi(1+p^\alpha) - (1+n)$.
Both terms on the right are even, so $L_-^*(n)$ is even.
If $n = 2^\beta \Pi p^\alpha$, all $p$ odd primes:
$L_-^*(n) = (1+2^\beta)\Pi(1+p^\alpha) - (1+n)$. The terms on the right are of opposite parity; and $L_-^*(n)$ is odd.

*Theorem A.* $L_-^*$ has no fixed points.

*Proof.* By Lemma 2, need only consider cases where parity of $n$ and $L_-^*(n)$ are the same. By Lemma 1, in these cases $L_-^*(n) < n$.

## 2. THE INEQUALITY $L_-^*(n) > n$

If $(L_-^*)^k(n) = n$, $k \geqslant 2$, then the images $L_-^*(n)$, $(L_-^*)^2(n)$, $\cdots$, $(L_-^*)^{k-1}$ are also fixed points of $(L_-^*)^k$. Thus fixed points of $(L_-^*)^k$, $k \geqslant 2$, arise in sets of $k$ distinct points. In each set of fixed points, there is at least one integer $m$ such that $L_-^*(m) > m$. The following propositions deal with the behavior of this inequality.

*Proposition 3.* If $k = nm$, $(n,m) = 1$, then $L_-^*(k) > L_-^*(n)L_-^*(m) + nL_-^*(m) + mL_-^*(n)$.

*Proof.*
$$L_-^*(k) = \sigma^*(k) - (1+k) = \sigma^*(n)\sigma^*(m) - (1+mn)$$
$$= [L_-^*(n) + (1+n)][L_-^*(m) + (1+m)] - (1+mn)$$
$$= L^*(n)L_-^*(m) + (1+n)L_-^*(m) + (1+m) + (1+m)$$
$$> L_-^*(n) + m + n$$
$$> L_-^*(n)L_-^*(m) + nL_-^*(m) + mL_-^*(n).$$

*Corollary 4.* If $k = nm$, $(n,m) = 1$, then
$$L_-^*(k) < L_-^*(m)L_-^*(n) + (1+m)L_-^*(n) + (1+n)L_-^*(m) + (1+m)(1+n).$$

For $k = 210 = 2\cdot3\cdot5\cdot7$, let $m = 6$, $n = 35$.
The inequality of the proposition is:
$$365 = L_-^*(210) > L_-^*(6)L_-^*(35) + 6L_-^*(35) + 35L_-^*(6) = 5\cdot12 + 5\cdot35 + 12\cdot6 = 307.$$
The corollary inequality, with these numbers is:
$$365 < 5\cdot12 + 5\cdot35 + 12\cdot6 = 307.$$
Relative primeness is necessary in the proposition. For $k = 90$, $m = 6$, $n = 15$, the required inequality is
$$80 > 5\cdot8 + 6\cdot8 + 15\cdot5 = 163$$
which is false.

*Proposition 5.* Let $m = p^\alpha n$, $(p,n) = 1$. If $L_-^*(n) > n$, then $L_-^*(m) > m$.

*Proof.*
$$L_-^*(n) > n \Rightarrow \sigma^*(n) - (1+n) > n \Rightarrow \sigma^*(n) - n > n$$
$$L_-^*(m) = \sigma^*(p^\alpha n) - (1+p^\alpha n) = (1+p^\alpha)\sigma^*(n) - 1 - p^\alpha n$$
$$= \sigma^*(n) - (1+n) + p^\alpha[\sigma^*(n) - n] + n = L_-^*(n) + p^\alpha[\sigma^*(n) - n] + n$$
$$> p^\alpha n + 2n > m.$$

If $m = p^{\alpha}n,\ (p,n) = p,$ the result does not necessarily follow.

$$L_-^*(30) = 41 > 30; \qquad L_-^*(60) = 59 < 60.$$

The inequality fails.

*Proposition 6.* Let $n = \Pi p^{\alpha}$. If $L_-^*(n) > n$, then

$$\Pi\left(1 + \frac{1}{p^{\alpha}}\right) > 2.$$

*Proof.* The inequality $\sigma^*(n) - (1+n) > n$ is also written as $\sigma^*(n) > 2n + 1$. Then

$$\frac{\sigma^*(n)}{n} > 2 + \frac{1}{n} > 2.$$

But

$$\frac{\sigma^*(n)}{n} = \frac{\Pi(1+p^{\alpha})}{\Pi p^{\alpha}} = \Pi\left(1 + \frac{1}{p^{\alpha}}\right).$$

*Corollary 7.* Let $n = \Pi p^{\alpha}$. If $L_-^*(n) > n$, then

$$\Pi\left(1 + \frac{1}{p}\right) > 2.$$

The results in Proposition 6 and Corollary 7 are necessary conditions but not sufficient. The inequalities are first satisfied by an integer $n$ with exponents $\alpha$ equal to 1. Among even integers, $n = 30 = 2\cdot 3\cdot 5$ is the smallest. $L_-^*(30) = 41$. Among odd integers, $n = 15015 = 3\cdot 5\cdot 7\cdot 11\cdot 13$ is the smallest. $L_-^*(15015) = 17240$.

*Corollary 8.* If $n$ is even and $L_-^*(n) > n$, then $n$ is divisible by at least three distinct primes.

*Corollary 9.* If $n$ is odd and $L_-^*(n) > n$, then $n$ is divisible by at least five distinct primes.

*Proposition 10.* For each natural number $n$, there is a natural number $t = t(n)$ such that for $k \geqslant t$,

$$L_-^*(n^k) < n^k.$$

*Proof.* Let $n = \Pi p^{\alpha}$; and $\underline{\underline{Q}}$, the rationals. The function $\theta : \underline{\underline{N}} \times \underline{\underline{N}} \to \underline{\underline{Q}}$ defined by

$$\theta(n,s) = \Pi\left[1 + \left(\frac{1}{p^{\alpha}}\right)^{s}\right]$$

is, for fixed $n$, a decreasing function of $s$ bounded below by 1. Let $t$ be the first integer such that

$$\theta(n,t) \leqslant 2.$$

*Proposition 11.* Let $m = p^{\alpha}n,\ (p,n) = 1$ with $L_-^*(n) < n$. If $L_-^*(m) > m$, then

$$\frac{2n}{2n - \sigma^*(n)} > p^{\alpha}.$$

*Proof.*

$$L_-^*(n) < n \Rightarrow \sigma^*(n) - (1+n) < n \Rightarrow \sigma^*(n) - 1 < 2n$$

$$L_-^*(m) > m \Rightarrow \sigma^*(p^{\alpha}n) - (1 + p^{\alpha}n) > p^{\alpha}n$$

$$\Rightarrow \sigma^*(n) - 1 > 2p^{\alpha}n - p^{\alpha}\sigma^*(n).$$

Then using the first inequality

$$2n > \sigma^*(n) - 1 > p^{\alpha}[2n - \sigma^*(n)] \quad \text{and} \quad \frac{2n}{2n - \sigma^*(n)} > p.$$

This proposition sets the bound on the multiples of a natural number $n$, $L_-^*(n) < n$, which enjoy the reverse inequality. For $n = 10$, $\sigma^*(n) = 18$,

$$\frac{2n}{2n - \sigma^*(n)} = \frac{20}{20 - 18} = \frac{20}{2} = 10.$$

The possible $p^{\alpha}$ are $3, 7, 3^2$, $L_-^*(30) = 41$. $L_-^*(70) = 73$; $L_-^*(90) = 89$.

Note that for 90,

$$\Pi\left(1+\frac{1}{p^{\alpha}}\right)=2.$$

## 3. THE COMPUTER SEARCH FOR FIXED POINTS OF ITERATES OF $L_-^*$

A computer search for natural numbers $n$ such that $L_-^*(n) > n$, $0 < n < 110{,}000$, was run on an IBM 370, model 135. For each such natural number $n$, the iterated values $(L_-^*)^k(n)$ were calculated, until 0 was reached. The program allowed fifty iterations. The values under the iterations were printed out. The process thus identifies any set of fixed points with an element less than, or equal to, 110,000.

Table 1 summarizes the results. There were no fixed points discovered. For all integers examined, iterations of $L_-^*$ eventually reached zero. For each $n$, the order of $n$ is the first integer $k$ such that $(L_-^*)^k(n) = 0$. For each value of the order, we list the first occurrence of the order and the frequency, or count, of the natural numbers with that order. The first natural number examined was 30; the last, 109,986. Note that there are no numbers of order 3 in the internal. Further the count of odd orders is relatively small. This can be explained, in part, by the few odd numbers under 110,000 satisfying $L_-^*(n) > n$. Recall that the first such is 15015. A total of 7697 numbers were examined.

It is desirable to develop upper and lower bounds for the first integer which is fixed under $(L_-^*)^2$.

**Table 1**

$L_-^*$

| Order | 1st | Frequency |
|---|---|---|
| 2 | 30 | 2203 |
| 3 | — | 0 |
| 4 | 66 | 1947 |
| 5 | 1596 | 10 |
| 6 | 294 | 1733 |
| 7 | 3290 | 38 |
| 8 | 854 | 1133 |
| 9 | 1190 | 46 |
| 10 | 4854 | 446 |
| 11 | 15890 | 20 |
| 12 | 14630 | 121 |
| 13 | 21945 | 8 |
| 14 | 38570 | 5 |
| 15 | 76670 | 4 |
| 16 | 104510 | 1 |
| 17 | 107030 | 1 |

## 4. THE FUNCTION $L_+^*$

In this section we examine $L_+^*$. For any natural number $n$, $L_+^*(n) = L_-^*(n) + 2$. So

$$L_-^*(n) > n \Rightarrow L_+^*(n) > n.$$

$$L_+^*(1) = 1; \quad L_+^*(2) = 2.$$

Thus $L_+^*$ has at least two fixed points.

Proofs of the following results parallel those above.

*Lemma 12.* For $n \in \underline{N}$, $L_+^*(n) = 2$ if, and only if, $n = p^{\alpha}$, $p$ a prime, $\alpha \geqslant 1$.

*Lemma 13.* For $n$ in $\underline{N}$, $L_+^*(n)$ has the same parity as $n$ if, and only if, $n = 2^{\alpha}$; $\alpha > 0$.

*Theorem B.* $L_+^*$ has exactly two fixed points, 1 and 2.

*Proposition 14.* If $k = mn$, $(m,n) = 1$ then $L_+^*(k) < L_+^*(m)L_+^*(n) + mL_+^*(n) + nL_+^*(m)$.

*Proposition 15.* Let $m = p^\alpha n$, $(p,n) = 1$. If $L_+^*(n) > n$, then $L_+^*(m) > m$.

*Proposition 16.* Let $n = \Pi p^\alpha$. If

$$\Pi\left(1 + \frac{1}{p^\alpha}\right) > 2,$$

then $L_+^*(n) > n$.

*Corollary 17.* Let $n = \Pi p^\alpha$. If

$$\Pi\left(1 + \frac{1}{p}\right) > 2,$$

then $L_+^*(n) > n$.

Recall that in Proposition 6 and Corollary 7, the condition

$$\Pi\left(1 + \frac{1}{p}\right) > \Pi\left(1 + \frac{1}{p^\alpha}\right) > 2$$

was necessary but not sufficient. Here it is sufficient but not necessary.

*Proposition 18.* For each natural number $n$, there is a natural number $t = t(n)$ such that for $k > t$, $L_+^*(n^k) < n^k$.

*Proof.* Using the notation of the proof of Proposition 10, it suffices to let $t$ be the first integer such that

$$\theta(n, t) \leqslant \frac{3}{2}.$$

*Proposition 19.* Let $m = p^\alpha n$, $(p,n) = 1$ with $L_+^*(n) < n$. If $L_+^*(m) > m$, then

$$\frac{2n}{2n - \sigma^*(n)} > p^n.$$

A computer search for natural numbers $n$ such that $L_+^*(n) > n$ was run, $0 < n < 100{,}000$. The iteration values were calculated and printed up to fifty iterations. The end value for iterations is 2 rather than 0. The search would have discovered any set of fixed points of an iterate of $L_+^*$ where one element of the set was less than, or equal to, 100,000. None were found. The results are in Table 2. The organization is as for Table 1.

**Table 2**

| Order | $L_+^*$ 1st | Frequency |
|---|---|---|
| 1 | 1,2 | 2 |
| 2 | 6 | 2020 |
| 3 | 82005 | 2 |
| 4 | 42 | 1274 |
| 5 | 498 | 27 |
| 6 | 78 | 1213 |
| 7 | 2530 | 144 |
| 8 | 402 | 1154 |
| 9 | 10650 | 72 |
| 10 | 1518 | 698 |
| 11 | 19635 | 19 |
| 12 | 2470 | 289 |
| 13 | 15015 | 2 |
| 14 | 10158 | 85 |
| 15 | – | 0 |
| 16 | 57030 | 15 |
| 17 | 84315 | 1 |

## REFERENCES

1. R. P. Jerrard and N. Temperley, "Almost Perfect Numbers," *Math Magazine,* 46 (1973), pp. 84–87.
2. M. Lal and A. Forbes, "A Note on Chowla's Function," *Math. Comp.,* 25 (1971), pp. 923–925. MR 45-6737.

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# FIBONACCI ASSOCIATION RESEARCH CONFERENCE

October 22, 1977

Host: MENLO COLLEGE
(El Camino Real)
Menlo Park, Calif.

| | |
|---|---|
| 9:30 | Opening Remarks: V. E. Hoggatt, Jr. |
| 9:40 to 10:20 | ARITHMETIC DIVISORS OF HIGHER ORDER<br>Krishna Alladi (UCLA) |
| 10:30 to 11:00 | FIBONACCI CHROMATOLOGY or HOW TO COLOR YOUR RABBIT<br>Marjorie Johnson (Wilcox High School) |
| 11:10 to 12:00 | PARTITIONS SUMS OF GENERALIZED PASCAL'S TRIANGLES<br>(A Second Report)<br>Claudia Smith (SJSU) or Verner E. Hoggatt, Jr. (SJSU) |
| 12:00 | LUNCH – to each his own |
| 1:30 to 2:10 | APPLICATIONS OF CERTAIN BASIC SEQUENCE CONVOLUTIONS TO FIBONACCI NUMBERS<br>Rodney Hansen (MSU, Bozeman, Montana) |
| 2:20 to 3:00 | GAMBLER'S RUIN AND FIBONACCI NUMBERS<br>Fred Stern (SJSU) |
| 3:10 to 3:30 | ENUMERATION OF CHESS GAME ENDINGS<br>George Ledin, Jr. (USF) |
| 3:40 to 4:20 | PRIMER ON STERN'S DIATOMIC SEQUENCE<br>Bob and Tina Giuli (SJSU) |