ON THE EVALUATION OF CERTAIN INFINITE SERIES BY ELLIPTIC FUNCTIONS

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1. INTRODUCTION

In this paper, we will obtain closed form expressions for certain series involving hyperbolic secants and cosecants, in terms of complete elliptic integrals of the first and second kind. By specializing, we will obtain closed form expressions for series involving the reciprocals of the well known Fibonacci and Lucas sequences, thereby indicating how similar series for related sequences may be evaluated. Also, we will derive some elegant symmetrical relationships, which enable numerical evaluation of such series with a high degree of precision.

2. REVIEW

We will begin by recalling some of the basic definitions and properties of Jacobian elliptic function theory which are relevant to the topic of this paper. The notation used will be that found in [1]; the formulas quoted in this section are also taken from [1], for the most part, or in some cases from [2], with revised notation.

$$u = u(\varphi, m) = \int_0^{\varphi} (1 - m\sin^2\theta)^{-1/2}\,d\theta. \tag{1}$$

The angle φ is called *amplitude,* and we write

$$\varphi = \operatorname{am} u. \tag{2}$$

In this paper, we will restrict φ to the two values 0 and $\pi/2$, and m to the open interval (0,1). Note that, in this domain of definition, u is a non-negative real number, and that $\lim_{m \to 1^-} u(\pi/2, m) = \infty$.

$$K = K(m) = u(\pi/2, m); \quad K' = K'(m) = u(\pi/2, 1-m) = K(1-m). \tag{3}$$

$$E = E(m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{1/2}\,d\theta; \qquad E' = E(1-m). \tag{4}$$

K and E are called the *complete elliptic integrals of the first and second kind,* respectively.

$$\operatorname{sn} u = \sin\varphi; \tag{5}$$

$$\operatorname{cn} u = \cos\varphi; \tag{6}$$

$$\operatorname{dn} u = (1 - m\sin^2\varphi)^{1/2}. \tag{7}$$

In (5)–(7) (as well as in the other nine Jacobian elliptic functions, which are derived from these, and not indicated here), if we wish to draw attention to the dependence of the function upon the parameter m, we write $\operatorname{sn}(u\,|\,m)$ for $\operatorname{sn} u$, etc.

For the values of φ with which we are concerned in this paper, we obtain the following relations:

$$\operatorname{sn} K = 1; \qquad \operatorname{cn} 0 = \operatorname{dn} 0 = 1; \qquad \operatorname{dn} K = (1-m)^{1/2}. \tag{8}$$

We observe from the definition of $K(m)$ that it is a monotonic increasing (continuous) mapping of (0,1) onto $(\pi/2, \infty)$; it then follows that the functions x and y defined by:

$$x = x(m) = \pi K'(m)/K(m), \quad \text{and} \quad y = y(m) = \pi K(m)/K'(m), \tag{9}$$

are one-to-one mappings of (0,1) onto $(0,\infty)$. (The notation introduced in (9) is not standard).

We also make the following definitions:

(10) $$q = \exp(-\pi K'/K) = e^{-x}; \qquad v = \pi u/2K.$$

In view of the preceding discussion, we see that $0 < q < 1$; moreover, for the two admissible values of φ which we allow, we obtain two possible triplets (u,v,φ), namely: $(0,0,0)$ and $(K, \pi/2, \pi/2)$.

So-called q-series expansions for the functions given in (3)–(7) exist, as well as for some related functions which we will consider, and these are simply listed below:

(11) $$\operatorname{sn} u = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1-q^{2n+1}} \sin(2n+1)v;$$

(12) $$\operatorname{cn} u = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \cos(2n+1)v;$$

(13) $$\operatorname{dn} u = \pi/2K + 2\pi/K \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos 2nv;$$

(14) $$(K/\pi)^2 \operatorname{dn}^2 u - (KE)/\pi^2 = 2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} \cos 2nv;$$

(15) $$\frac{4}{3}(2-m)(K/\pi)^2 - 4(KE)/\pi^2 + 1/3 = 8 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}};$$

(16) $$1 - 4(KE)/\pi^2 = 8 \sum_{n=1}^{\infty} \frac{(-1)^n nq^{2n}}{1-q^{2n}};$$

(17) $$-1/16 \log(1-m) = \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(2n-1)(1-q^{4n-2})}.$$

3. CLOSED FORMS

If, in (11)–(14), we substitute the special values of u and v indicated in the paragraph following (10), we eliminate the trigonometric terms occurring in these identities. We may also make the substitution indicated in (10), and if appropriate, extend the summation variable over all integral values. The result of these manipulations is the following list of identities:

(18) $$2 \sum_{n=1}^{\infty} (-1)^{n-1} \operatorname{csch}(n-\tfrac{1}{2})x = \sum_{n=-\infty}^{\infty} \operatorname{sech}(n-\tfrac{1}{2})x = 2Km^{1/2}/\pi;$$

(19) $$\sum_{n=-\infty}^{\infty} \operatorname{sech} nx = 2K/\pi;$$

(20) $$\sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} nx = 2K(1-m)^{1/2}/\pi;$$

(21) $$\sum_{n=1}^{\infty} n \operatorname{csch} nx = K(K-E)/\pi^2;$$

(22)
$$\sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} nx = KE/\pi^2 - (1-m)(K/\pi)^2 .$$

Since $0 < q < 1$, the following series manipulations are valid:

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nq^{(2j-1)n} = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} nq^{(2j-1)n} = \sum_{j=1}^{\infty} \frac{q^{2j-1}}{(1-q^{2j-1})^2};$$

that is,

$$\sum_{n=1}^{\infty} n \operatorname{csch} nx = \tfrac{1}{2} \sum_{n=1}^{\infty} \operatorname{csch}^2 (n-\tfrac{1}{2})x .$$

In a similar manner, we may prove the following identities:

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} nx = \tfrac{1}{2} \sum_{n=1}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})x ;$$

$$\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} = \tfrac{1}{4} \sum_{n=1}^{\infty} \operatorname{csch}^2 nx ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} nq^{2n}}{1-q^{2n}} = \tfrac{1}{4} \sum_{n=1}^{\infty} \operatorname{sech}^2 nx .$$

Incorporating these results into (15), (16), (21) and (22), we obtain:

(23)
$$\sum_{n=-\infty}^{\infty} \operatorname{sech}^2 nx = 4KE/\pi^2 ;$$

(24)
$$\sum_{n=1}^{\infty} \operatorname{csch}^2 nx = 1/6 + 2/3\,(2-m)(K/\pi)^2 - 2KE/\pi^2 ;$$

(25)
$$\sum_{n=1}^{\infty} 2n \operatorname{csch} nx = \sum_{n=1}^{\infty} \operatorname{csch}^2 (n-\tfrac{1}{2})x = 2K(K-E)/\pi^2 ;$$

(26)
$$\sum_{n=1}^{\infty} 2(-1)^{n-1} n \operatorname{csch} nx = \sum_{n=1}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})x = 2KE/\pi^2 - 2(1-m)(K/\pi)^2 .$$

Finally, equation (17) may be recast as follows:

(27)
$$\sum_{n=1}^{\infty} \frac{\operatorname{csch} (2n-1)x}{2n-1} = -1/8 \log (1-m).$$

The results with which we are interested are (18)–(20) and (23)–(27). These are all identities in the implicit parameter m. However, we may also view them as identities in the summand parameter x, since m, and therefore $K(m)$, $K'(m)$ and $E(m)$ are uniquely determined by (9), for any given positive x. In this sense, then, (18)–(20) and (23)–(27) represent *closed form* expressions for the indicated series, where the sums are expressed as implicit functions of x.

As a matter of interest, we include below two identities free of terms involving m, derived by inspection of (18), (19) and (23)–(26):

$$(28)\quad \sum_{n=1}^{\infty} (\operatorname{sech}^2 nx + \operatorname{csch}^2 nx) = -1/3 + 1/3\left(\sum_{n=-\infty}^{\infty} \operatorname{sech} nx\right)^2 - 1/6\left(\sum_{n=-\infty}^{\infty} \operatorname{sech}(n-\tfrac{1}{2})x\right)^2, \ \forall x \neq 0.$$

$$(29)\quad \sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 (n-\tfrac{1}{2})x + \operatorname{csch}^2 (n-\tfrac{1}{2})x) = \left(\sum_{n=-\infty}^{\infty} \operatorname{sech}(n-\tfrac{1}{2})x\right)^2, \qquad \forall x \neq 0.$$

4. APPLICATIONS TO SERIES INVOLVING RECIPROCALS OF FIBONACCI AND LUCAS NUMBERS

Consider the sequence $\{U_n\}_0^\infty$ of non-negative integers defined by the recursion:

$$(30)\qquad U_{n+2} = aU_{n+1} + bU_n, \qquad n = 0, 1, 2, \cdots,$$

where a, b, U_0 and U_1 are given non-negative integers, with a and b not both zero, U_0 and U_1 not both zero. It is known from the theory of linear difference equations that an explicit formula for U_n exists, given by:

$$(31)\qquad U_n = U_1 G_n + bU_0 G_{n-1}, \qquad n = 1, 2, 3, \cdots,$$

where

$$(32)\qquad G_n = \frac{r^n - s^n}{r - s}, \qquad n = 0, 1, 2, \cdots,$$

and

$$(33)\qquad r = \tfrac{1}{2}(a + \sqrt{a^2 + 4b}), \qquad s = \tfrac{1}{2}(a - \sqrt{a^2 + 4b}).$$

Note that $r > 0$. If, in particular, $b = 1$, and if we let $L = \log r$, then G_n takes a form which is of interest to the topic of this paper. Specifically,

$$(34)\qquad G_{2n} = \frac{2}{\sqrt{a^2+4}} \sinh 2nL, \qquad G_{2n+1} = \frac{2}{\sqrt{a^2+4}} \cosh (2n+1)L, \quad n = 0, 1, 2, \cdots.$$

Thus, for certain special values of a, U_0 and U_1, we see that the identities of the previous section may be used to obtain closed form expressions for series involving the reciprocals of our particular sequence $\{U_n\}$.

We illustrate with a specific example, by taking $a = b = 1$. Then let

$$(35)\qquad \alpha = r = \tfrac{1}{2}(1 + \sqrt{5}), \quad \beta = s = \tfrac{1}{2}(1 - \sqrt{5}), \quad \lambda = L = \log \alpha.$$

The sequence $\{G_n\}$ then becomes the familiar *Fibonacci* sequence $\{F_n\}$; using (34), we see that the general term of this sequence is given by:

$$(36)\qquad F_{2n} = 2/\sqrt{5}\,\sinh 2n\lambda, \qquad F_{2n+1} = 2/\sqrt{5}\,\cosh (2n+1)\lambda, \qquad n = 0, 1, \cdots.$$

If we take $U_0 = 0$, $U_1 = 1$ as initial values, then the sequence $\{U_n\}$ coincides with $\{F_n\}$. If we take $U_0 = 2$, $U_1 = 1$, the resulting sequence is the *Lucas* sequence $\{L_n\}$, whose general term is as follows:

$$(37)\qquad L_{2n} = 2\cosh 2n\lambda, \qquad L_{2n+1} = 2\sinh (2n+1)\lambda, \qquad n = 0, 1, \cdots.$$

If, in definition (9), we let $x = 2\lambda$, this determines a unique constant μ, such that $0 < \mu < 1$, and

$$(38)\qquad \pi K'(\mu)/K(\mu) = 2\lambda.$$

Also, let $\rho = K(\mu)/\pi$, $\sigma = E(\mu)/\pi$. For this particular value of x, we may then use (18)–(20), (23)–(27) and (36)–(38) to obtain the following closed-form expressions:

$$(39)\qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n-1}} = \tfrac{1}{2}\rho\sqrt{\mu};$$

$$(40)\qquad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = \tfrac{1}{2}\rho\sqrt{5\mu};$$

(41) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = \tfrac{1}{2}\rho - \tfrac{1}{4};$$

(42) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n}} = \tfrac{1}{4} - \tfrac{1}{2}\rho\sqrt{1-\mu};$$

(43) $$\sum_{n=1}^{\infty} \left(\frac{1}{L_{2n}}\right)^2 = \tfrac{1}{2}\rho\sigma - 1/8;$$

(44) $$\sum_{n=1}^{\infty} \left(\frac{1}{F_{2n}}\right)^2 = \frac{5}{24} + \frac{5}{6}(2-\mu)\rho^2 - \frac{1}{2}\rho\sigma;$$

(45) $$\sum_{n=1}^{\infty} \frac{n}{F_{2n}} = \tfrac{1}{2}\sqrt{5}\,\rho(\rho-\sigma);$$

(46) $$\sum_{n=1}^{\infty} \left(\frac{1}{L_{2n-1}}\right)^2 = \tfrac{1}{2}\rho(\rho-\sigma);$$

(47) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{F_{2n}} = \tfrac{1}{2}\sqrt{5}(\rho\sigma - (1-\mu)\rho^2);$$

(48) $$\sum_{n=1}^{\infty} \left(\frac{1}{F_{2n-1}}\right)^2 = \frac{5}{2}(\rho\sigma - (1-\mu)\rho^2);$$

(49) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)F_{4n-2}} = -\sqrt{5}/16\ \log(1-\mu).$$

Since all the series in (39)–(49) are absolutely convergent, we may obtain other formulas by combinations of the foregoing expressions. For example, if we alternately add and subtract (41) and (42), we obtain:

(50) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}} = \tfrac{1}{4}(1-\sqrt{1-\mu})\rho,$$

and

(51) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n}} = \tfrac{1}{4}(1+\sqrt{1-\mu})\rho - \tfrac{1}{4}\ .$$

A similar process on (45) and (47) yields the pair of identities:

(52) $$\sum_{n=1}^{\infty} \frac{2n-1}{F_{4n-2}} = \tfrac{1}{4}\sqrt{5}\,\mu\rho^2\ ;$$

(53) $$\sum_{n=1}^{\infty} \frac{2n}{F_{4n}} = \tfrac{1}{4}\sqrt{5}\left\{(2-\mu)\rho^2 - 2\rho\sigma\right\}.$$

Adding (43) and (46) yields:

(54) $$\sum_{n=1}^{\infty} \frac{1}{L_n^2} = \tfrac{1}{2}\rho^2 - 1/8.$$

Adding (44) and (48) yields:

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2} = \frac{5}{24} + \frac{5}{6}(2\mu - 1)\rho^2. \tag{55}$$

Inspection of the preceding list of closed form expressions yields a variety of interesting identities, some of which are shown below:

$$3\sum_{n=1}^{\infty} \frac{1}{F_n^2} + 5\sum_{n=1}^{\infty} \frac{1}{L_n^2} = 4\left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2 = 80\left(\sum_{n=1}^{\infty} \frac{1}{L_{4n}}\right)\left(\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}}\right) = 5\mu\rho^2; \tag{56}$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} + 5\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = 2\left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2 = \frac{5}{2}\mu\rho^2. \tag{57}$$

The Lucas sequence may be extended to negative indices, by the following definition, which is consistent with the definition in (37):

$$L_{-n} = (-1)^n L_n, \qquad n = 0, 1, 2, \cdots. \tag{58}$$

Using (58), we obtain the following elegant identity:

$$\sum_{n=-\infty}^{\infty} \frac{1}{L_n^2} = \left(\sum_{n=-\infty}^{\infty} \frac{1}{L_{2n}}\right)^2 = \rho^2. \tag{59}$$

Note

$$\sum_{n=-\infty}^{\infty} \frac{1}{L_{2n}} = \rho.$$

One more identity is worth including, namely:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)L_{2n-1}} = 1/8 \log\left(\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}\right). \tag{60}$$

This does not follow from any previous identity in this section, though similar to (49). The proof of (60) depends upon a general theorem about elliptic functions, which properly does not belong in this section; it is nevertheless instructive to include it here, illustrating how the basic identities in (18)–(20) and (23)–(27) may be made to yield other identities not previously covered.

Theorem. Suppose $2K'(m_1)/K(m_1) = K'(m_2)/K(m_2)$.

Then:

(a) $$K(m_1) = (1+\sqrt{m_2})K(m_2);$$

(b) $$K'(m_1) = \tfrac{1}{2}(1+\sqrt{m_2})K'(m_2);$$

(c) $$m_1 = 1 - \left(\frac{1-\sqrt{m_2}}{1+\sqrt{m_2}}\right)^2 = \frac{4\sqrt{m_2}}{(1+\sqrt{m_2})^2};$$

(d) $$E'(m_1) = \frac{E'(m_2) + \sqrt{m_2}\,K'(m_2)}{1+\sqrt{m_2}};$$

(e) $$E(m_1) = \frac{2E(m_2) - (1-m_2)K(m_2)}{1+\sqrt{m_2}}.$$

Proof of (a). Let $x = \pi K'(m_1)/K(m_1)$. Observing that the series in (18) and (19) are absolutely convergent (this is actually true for all of the series in (18)–(20), (23)–(27)), provided, of course, x is real and non-zero, the following manipulation is valid:

$$\sum_{n=-\infty}^{\infty} \operatorname{sech} 2nx + \sum_{n=-\infty}^{\infty} \operatorname{sech} (2n-1)x = \sum_{n=-\infty}^{\infty} \operatorname{sech} nx .$$

Using (18), (19) and the hypothesis, this is equivalent to the following relation:

$$\frac{2}{\pi} K(m_2) + \frac{2}{\pi} m_2^{1/2} K(m_2) = \frac{2}{\pi} K(m_1).$$

This implies (a).

Proof of (b): An immediate consequence of (a) and the hypothesis.

Proof of (c): The following is Formula 17.3.29 in [1], slightly modified:

$$K(m) = \frac{2}{1+\sqrt{1-m}} K\left\{\left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right\} .$$

Replacing m by $1-m_2$ yields:

$$K'(m_2) = \frac{2}{1+\sqrt{m_2}} K\left\{\left(\frac{1-\sqrt{m_2}}{1+\sqrt{m_2}}\right)^2\right\} .$$

Substituting this result into (b) yields:

$$K'(m_1) = K(1-m_1) = K\left\{\left(\frac{1-\sqrt{m_2}}{1+\sqrt{m_2}}\right)^2\right\} .$$

This result and the fact that K is a one-to-one function on (0,1) imply (c).

Proof of (d): The following is Formula 17.3.30 in [1], slightly modified:

$$E(m) = (1+\sqrt{1-m})E\left\{\left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right\} - \frac{2\sqrt{1-m}}{1+\sqrt{1-m}} K\left\{\left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right\} .$$

Replacing m by $1-m_2$ and incorporating the results of (b) and (c) yields:

$$E'(m_2) = (1+\sqrt{m_2})E'(m_1) - \sqrt{m_2}\,K'(m_2).$$

Rearrangement yields (d).

Proof of (e): The following is the famous relation due to Legendre:

$$EK' + E'K - KK' = \pi/2,$$

for any (implicit) parameter m. Letting $m = m_1$ and substituting the results of (a), (b) and (d) yields (e). This completes the proof of the theorem.

If the constant μ_1 is defined by:

$$\pi K'(\mu_1)/K(\mu_1) = \lambda,$$

it follows from part (c) of the preceding theorem that μ_1 is related to μ by the following identity:

$$\mu_1 = 1 - \left(\frac{1-\sqrt{\mu}}{1+\sqrt{\mu}}\right)^2 .$$

Equation (60) then follows from this last result, by substituting $x = \lambda$ in (27) and using (37). This same substitution in the other identities of Section 3, however, results either in series which have already been treated (by decomposition into even and odd terms), or in series whose terms contain irrational numbers. Therefore, if we are interested only in obtaining closed forms for series of *rational* numbers, identity (27) is the only identity in Section 3 which yields an "interesting" result for $x = \lambda$. It would therefore appear that the theorem we have proved has very limited applicability. This is not the case, however, for if we solve for the functions of m_2 in terms of the functions of m_1, we obtain formulas for other "interesting" series not previously treated, in terms of the original parameter m_1. Theoretically, this process may be continued indefinitely, but the closed forms thereby obtained will become increasingly cumbersome at each step. To illustrate, we set $m_1 = \mu$ in the theorem of this section, and define μ'' by the relation:

$$\pi K'(\mu'')/K(\mu'') = 4\lambda \; ;$$

hence, μ'' plays the role of m_2 in the theorem. Also, let

$$\rho'' = K(\mu'')/\pi \qquad \text{and} \qquad \sigma'' = E(\mu'')/\pi \, .$$

Using the theorem, we may solve for the "double-primed" functions in terms of the unprimed functions, and obtain the following results:

(61) $$\sqrt{\mu''} = (1-\sqrt{1-\mu})/(1+\sqrt{1-\mu}\,; \qquad 1-\mu'' = \frac{4\sqrt{1-\mu}}{(1+\sqrt{1-\mu})^2} \; ;$$

(62) $$\rho'' = \tfrac{1}{2}(1+\sqrt{1-\mu})\rho\,; \qquad \sigma'' = (\sigma+\rho\sqrt{1-\mu})/(1+\sqrt{1-\mu}) \; .$$

If we substitute $x = 4\lambda$ in (18) and apply (36) and (37), we obtain the formulas:

$$\frac{4}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{4n-2}} = 4 \sum_{n=1}^{\infty} \frac{1}{L_{4n-2}} = 2\rho''\sqrt{\mu''} \; .$$

Now using the results of (61) and (62), we obtain the identities:

(63) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{4n-2}} = \tfrac{1}{4}\sqrt{5}\,(1-\sqrt{1-\mu})\rho \; ;$$

(64) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}} = \tfrac{1}{4}(1-\sqrt{1-\mu})\rho \; .$$

Similarly, we may derive the following identities from the general ones of Section 3, by means of the same substitutions:

(65) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n}} = \tfrac{1}{4}(1+\sqrt{1-\mu})\rho - \tfrac{1}{4} \; ;$$

(66) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{4n}} = \tfrac{1}{4} - \tfrac{1}{2}(1-\mu)^{1/4}\rho\,;$$

(67) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n}^2} = \tfrac{1}{4}(\rho\sigma+\rho^2\sqrt{1-\mu}) - \frac{1}{8} \; ;$$

(68) $$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2} = \frac{5}{24}\left\{1+(2-\mu)\rho^2 - 6\rho\sigma\right\} \; ;$$

(69) $$\sum_{n=1}^{\infty} \frac{4n}{F_{4n}} = \tfrac{1}{2}\sqrt{5}\left\{(2-\mu)\rho^2 - 2\rho\sigma\right\} \; ;$$

(70) $$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^2} = \frac{5}{8}\left\{(2-\mu)\rho^2 - 2\rho\sigma\right\} \; ;$$

(71) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4n}{F_{4n}} = \sqrt{5}\,(\rho\sigma - \rho^2\sqrt{1-\mu}\,)\,;$$

(72) $$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}^2} = \tfrac{1}{4}(\rho\sigma - \rho^2\sqrt{1-\mu}) \quad ;$$

(73) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)F_{8n-4}} = \frac{\sqrt{5}}{8} \log\left\{\frac{(1-\mu)^{1/4} + (1-\mu)^{-1/4}}{2}\right\} \quad .$$

Observe that (64) and (65) were previously derived, as indicated in (50) and (51), by a different method. Appropriate combinations of (64)–(72) yield the following identities (note that (78) and (79) were previously derived, as indicated in (43) and (44)):

(74) $$\sum_{n=1}^{\infty} \frac{1}{L_{8n}} = \frac{1}{8}\left\{1 + (1-\mu)^{1/4}\right\}^2 \rho - \tfrac{1}{4} ;$$

(75) $$\sum_{n=1}^{\infty} \frac{1}{L_{8n-4}} = \frac{1}{8}\left\{1 - (1-\mu)^{1/4}\right\}^2 \rho \; ;$$

(76) $$\sum_{n=1}^{\infty} \frac{8n}{F_{8n}} = \tfrac{1}{4}\sqrt{5}\left\{(1+\sqrt{1-\mu})^2\rho^2 - 4\rho\sigma\right\} \; ;$$

(77) $$\sum_{n=1}^{\infty} \frac{8n-4}{F_{8n-4}} = \tfrac{1}{4}\sqrt{5}(1-\sqrt{1-\mu})^2\rho^2 \; ;$$

(78) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = \tfrac{1}{2}\rho\sigma - 1/8 \; ;$$

(79) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2} = \frac{5}{24}\left\{1 + 4(2-\mu)\rho^2 - 12\rho\sigma\right\} .$$

By letting $x = 8\lambda$ in (18)–(20), (23)–(27), and again using the theorem of this section, we may derive yet another set of identities, involving the reciprocals of Fibonacci and Lucas numbers of indices $8n$ or $8n-4$ (except for the identity derived from (27), which involves F_{16n-8}); the closed forms thereby derived are again functions of the three basic constants μ, ρ and σ, albeit more complicated functions. Continuing in this fashion, we may, in theory, obtain closed forms for series involving the reciprocals of Fibonacci and Lucas numbers, where their indices have one of the two forms: $2^k n$ or $2^k(2n-1)$. Note, however, that conspicuously absent from the compendium of identities in this section are formulas for the series:

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}} \quad .$$

It is seen, from (36) and (37), that these, in turn, depend on an evaluation of the series

$$\sum_{n=1}^{\infty} \operatorname{csch} nx ,$$

which is absent in Section 3. Such an evaluation does not appear to be provided by the elliptic function theory, however, and is, in fact, the subject of a separate section of this paper.

Mention should be made of recent papers by Greig and Gould ([5] and [6]), where elementary techniques are used to obtain approximations to the series

$$\sum_{n=1}^{\infty} \frac{1}{F_n},$$

and to more general series. The most significant result to the topic of this paper appears in [5] and may be expressed in the following form:

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{n=0}^{\infty} \left(\frac{1+F_{2n}}{F_{2n+1}} + \frac{2}{F_{4n+2}} + \beta \right).$$

This formula, however, does not yield a closed form, but only a rearrangement, of the terms in

$$\sum_{n=1}^{\infty} \frac{1}{F_n},$$

albeit one which yields fairly rapid convergence.

It is clear how the formulas of this section may be extended to other sequences (U_n) of the type discussed in the beginning of this section. It is not the aim of the author to obtain an indefinite number of identities such as are listed in this section, but rather to indicate the methods by which one may proceed in so doing.

5. SYMMETRICAL RELATIONSHIPS

Although the formulas of Section 3 (and their applications in Section 4) provide closed forms for the indicated series, they are not very satisfactory from the point of view of numerical evaluation; manual computations of m (from (9), with given x), and of $K(m)$ and $E(m)$, even with the help of tables of elliptic integrals and related tables, can be quite cumbersome, and in any event cannot exceed the accuracy of the tables. There is a much more satisfactory approach, fortunately, which enables the computation of m, K and E with a high degree of precision and a minimum of effort.

Recall the definitions of x and y given in (9), and note that $xy = \pi^2$. Note also that all of the Section 3 formulas are valid if x is replaced by y, m replaced by $(1-m)$, K replaced by K', and E replaced by E' (see (3) and (4) for definitions of K' and E'). However, K, K', E and E' are not independent of each other, but rather satisfy the relations:

(80) $$K' = Kx/\pi$$

(a restatement of (9)), and

(81) $$E' = \pi/2K + x/\pi \cdot (K - E)$$

(a restatement of Legendre's relation, incorporating the result of (80); see proof of part (e) of Theorem in Section 4).

By means of (80) and (81), we may express the formulas in Section 3 as functions of y, with closed forms in terms of m, K and E. If we then equate these expressions with the original functions of x, we obtain relations between functions of x and functions of y, which display a symmetry of some sort. We illustrate this method by deriving the following symmetrical relation:

(82) $$|x|^{1/2} \sum_{n=-\infty}^{\infty} \operatorname{sech} nx = |y|^{1/2} \sum_{n=-\infty}^{\infty} \operatorname{sech} ny, \ \forall \text{ real } x,y \text{ such that } xy = \pi^2.$$

The proof of (82) follows from (19) and (80):

$$\sum_{n=-\infty}^{\infty} \operatorname{sech} ny = 2K'/\pi = 2Kx/\pi^2 = 2K/y = \pi/y \sum_{n=-\infty}^{\infty} \operatorname{sech} nx = (x/y)^{1/2} \sum_{n=-\infty}^{\infty} \operatorname{sech} nx,$$

provided x,y are real positive numbers such that $xy = \pi^2$. Note, however, that this result is independent of elliptic functions and is equally valid if x and y are both negative, because sech is an even function. This establishes (82).

An asymmetrical relation is obtained by applying this method to (18) and (20), which again yields a result which is independent of m, namely:

$$|x|^{1/2}\sum_{n=-\infty}^{\infty}\operatorname{sech}(n-\tfrac12)x = |y|^{1/2}\sum_{n=-\infty}^{\infty}(-1)^n\operatorname{sech} ny \quad (x,y \text{ real}, xy=\pi^2). \tag{83}$$

Similarly, we may derive the following formulas, where in all cases, x and y are arbitrary real numbers such that $xy=\pi^2$:

$$\sum_{n=1}^{\infty}(-1)^{n-1}(nx\operatorname{csch} nx + ny\operatorname{csch} ny) = \tfrac12\ ; \tag{84}$$

$$\sum_{n=1}^{\infty}\left\{|x|\operatorname{sech}^2(n-\tfrac12)x + |y|\operatorname{sech}^2(n-\tfrac12)y\right\} = 1\ ; \tag{85}$$

$$\sum_{n=-\infty}^{\infty}|x|\operatorname{sech}^2 nx = 2+4\sum_{n=1}^{\infty}ny\operatorname{csch} ny = 2+\sum_{n=1}^{\infty}2|y|\operatorname{csch}^2(n-\tfrac12)y\ ; \tag{86}$$

$$\sum_{n=1}^{\infty}\left\{|x|\operatorname{csch}^2 nx + |y|\operatorname{csch}^2 ny\right\} = \frac{|x+y|}{6} - 1. \tag{87}$$

Aside from whatever elegance equations (82)–(87) possess, they are quite useful for numerical computations, for we may choose x in such a way that the series involving y converges with extreme rapidity. To see this better, we convert (82)–(87) to the forms which are more suitable for numerical computation, valid $\forall$ real $x \neq 0$:

$$\sum_{n=-\infty}^{\infty}\operatorname{sech} nx = \frac{\pi}{|x|}\sum_{n=-\infty}^{\infty}\operatorname{sech} n\pi^2/x\ ; \tag{88}$$

$$\sum_{n=-\infty}^{\infty}\operatorname{sech}(n-\tfrac12)x = \frac{\pi}{|x|}\sum_{n=-\infty}^{\infty}(-1)^n\operatorname{sech} n\pi^2/x\ ; \tag{89}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\operatorname{csch} nx = \frac{1}{2x} - \frac{\pi^2}{x^2}\sum_{n=1}^{\infty}(-1)^{n-1}n\operatorname{csch} n\pi^2/x\ ; \tag{90}$$

$$\sum_{n=1}^{\infty}\operatorname{sech}^2(n-\tfrac12)x = \frac{1}{|x|} - \frac{\pi^2}{x^2}\sum_{n=1}^{\infty}\operatorname{sech}^2(n-\tfrac12)\pi^2/x\ ; \tag{91}$$

$$\sum_{n=-\infty}^{\infty}\operatorname{sech}^2 nx = \frac{2}{|x|} + \frac{4\pi^2}{x^2}\sum_{n=1}^{\infty}n\operatorname{csch} n\pi^2/|x| = \frac{2}{|x|} + \frac{2\pi^2}{x^2}\sum_{n=1}^{\infty}\operatorname{csch}^2(n-\tfrac12)\pi^2/x\ ; \tag{92}$$

$$\sum_{n=1}^{\infty}\operatorname{csch}^2 nx = \frac16 - \frac{1}{|x|} + \frac{\pi^2}{6x^2} - \frac{\pi^2}{x^2}\sum_{n=1}^{\infty}\operatorname{csch}^2 n\pi^2/x\ . \tag{93}$$

By choosing $0 < |x| \leq \pi$, the convergence of the series in the right members of (88)–(93) is at least as rapid as that which occurs when $|x| = |y| = \pi$, which is itself fairly rapid. If we require $|x| > \pi$, we may then reverse the roles of x and y in (88)–(93), and still obtain rapid convergence, using the series in the *left* members to evaluate the required series.

6. NUMERICAL EVALUATION OF SERIES INVOLVING RECIPROCALS OF FIBONACCI AND LUCAS NUMBERS

In this section, we will apply the results of the previous section toward numerical evaluation of the constants μ, ρ and σ defined by (38). We first need to compute $\lambda = \log\{\tfrac{1}{2}(1+\sqrt{5})\}$. The computations indicated in this section were performed manually, with the help of tables found in [1]. In all cases, the accuracy does not exceed 15 significant digits. An electronic computer would attain far greater accuracy.

(94) $$\lambda = .48121\ 18250\ 59603, \text{ approximately.}$$

Substituting $x = 2\lambda$ (or $x = \pi^2/2\lambda$, where appropriate) in (88)–(93) yields, among others, the following identities:

(95) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = -\tfrac{1}{4} + \pi/8\lambda \sum_{n=-\infty}^{\infty} \operatorname{sech} ny, \quad \text{where } y = \pi^2/2\lambda,$$

(96) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = \frac{\pi\sqrt{5}}{8\lambda} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} ny;$$

(97) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} = \frac{5}{8\lambda} - \frac{5\pi^2}{32\lambda^2} \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})y;$$

(98) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2} = \frac{5}{24} - \frac{5}{8\lambda} + \frac{5\pi^2}{96\lambda^2} - \frac{5\pi^2}{16\lambda^2} \sum_{n=1}^{\infty} \operatorname{csch}^2 ny;$$

(99) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = -\frac{1}{8\lambda} + \frac{\pi^2}{32\lambda^2} \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 ny;$$

(100) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = -\frac{1}{8} + \frac{1}{8\lambda} + \frac{\pi^2}{32\lambda^2} \sum_{n=-\infty}^{\infty} \operatorname{csch}^2 (n-\tfrac{1}{2})y;$$

Adding (97) and (98) yields:

(101) $$\sum_{n=1}^{\infty} \frac{1}{F_n^2} = \frac{5}{24} - \frac{5\pi^2}{48\lambda^2} - \frac{5\pi^2}{16\lambda^2} \sum_{n=1}^{\infty} (\operatorname{sech}^2 (n-\tfrac{1}{2})y + \operatorname{csch}^2 ny).$$

Adding (99) and (100) yields:

(102) $$\sum_{n=1}^{\infty} \frac{1}{L_n^2} = -\frac{1}{8} + \frac{\pi^2}{32\lambda^2} \sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny + \operatorname{csch}^2 (n-\tfrac{1}{2})y).$$

If we now compare the results of (41) and (95), we obtain:

(103) $$\rho = \frac{\pi}{4\lambda} \sum_{n=-\infty}^{\infty} \operatorname{sech} ny.$$

Comparing (40) and (96) yields:

(104) $$\rho\sqrt{\mu} = \frac{\pi}{4\lambda} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} ny,$$

from which it follows that

(105) $$\sqrt{\mu} = \left(\sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sech} ny \right) \div \left(\sum_{n=-\infty}^{\infty} \operatorname{sech} ny \right).$$

The values of ρ^2 and μ can be obtained by squaring both sides of (103) and (105), respectively. An alternative approach is indicated below. If we compare (54) and (102), we obtain:

(106) $$\rho^2 = \frac{\pi^2}{16\lambda^2} \sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny + \operatorname{csch}^2 (n - \tfrac{1}{2})y).$$

Combining (97) and (99) as indicated in (57) and comparing the results, we obtain:

(107) $$\mu\rho^2 = \frac{\pi^2}{16\lambda^2} \sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny - \operatorname{sech}^2 (n - \tfrac{1}{2})y).$$

It follows from (106) and (107) that we have:

(108) $$\mu = \left(\sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny - \operatorname{sech}^2 (n - \tfrac{1}{2})y) \right) \div \left(\sum_{n=-\infty}^{\infty} (\operatorname{sech}^2 ny + \operatorname{csch}^2 (n - \tfrac{1}{2})y) \right).$$

Again, the computation of $\rho\sqrt{1-\mu}$ may be accomplished from the values of ρ and μ obtained in (103) and (108); a somewhat more accurate result is obtained, however, if we combine the results of (37), (42) and (83), which yields:

(109) $$\rho\sqrt{1-\mu} = \frac{\pi}{4\lambda} \sum_{n=-\infty}^{\infty} \operatorname{sech} (n - \tfrac{1}{2})y.$$

In the closed form expressions occurring in Section 4, we observe that the constant σ always appears multiplied by ρ; therefore, we will indicate the numerical computation of $\rho\sigma$, rather than of σ itself. This is most easily accomplished by combining the results of (43) and (100), which yields:

(110) $$\rho\sigma = \frac{1}{4\lambda} + \frac{\pi^2}{16\lambda^2} \sum_{n=-\infty}^{\infty} \operatorname{csch}^2 (n - \tfrac{1}{2})y.$$

Superficially, it would appear that the identities in (103)–(110) are very unwieldy for computational purposes. However, as mentioned previously, the infinite series in the right members of (103)–(110) converge quite rapidly; thus, at most eight terms of the series need be included to guarantee an accuracy in the result of 15 significant digits! Moreover, since the summand terms are symmetrical about the value $n = 0$, only four terms of the series, at most, need be computed for 15-digit accuracy! A summary of the computations is appended; indicated in Appendix II are the *computed* values of the series occurring in Section 4, using the constants indicated in Appendix I. As a check on the computations, the actual summations were performed by the author with the aid of a desk calculator, and all results checked with those indicated in Appendix II, to 15 significant digits! It should be emphasized that the values in Appendix II were obtained *without* performing any actual summations.

7. CONCLUSION

As mentioned previously, the series

$$\sum_{n=1}^{\infty} \operatorname{csch} nx,$$

(x real and non-zero), apparently cannot be evaluated by elliptic functions. However, the following formula in terms of Lambert functions exists:

(111) $$\sum_{n=1}^{\infty} \operatorname{csch} nx = 2\{\pounds(e^{-x}) - \pounds(e^{-2x})\}, \quad (x > 0),$$

where

(112) $$\pounds(q) = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}, \quad |q| < 1,$$

is the *Lambert function.*

By decomposing (111) into even and odd-subscript terms, we may deduce the following formulas:

(113) $$\sum_{n=1}^{\infty} \operatorname{csch} 2nx = 2\{\pounds(e^{-2x}) - \pounds(e^{-4x})\};$$

and

(114) $$\sum_{n=1}^{\infty} \operatorname{csch}(2n-1)x = 2\{\pounds(e^{-x}) - 2\pounds(e^{-2x}) + \pounds(e^{-4x})\}, \text{ where } x > 0.$$

In particular, setting $x = \lambda$ in (113)–(114) and employing (36)–(37), we obtain the following formulas:

(115) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5}\,\{\pounds(\beta^2) - \pounds(\beta^4)\},$$

and

(116) $$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}} = \pounds(-\beta) - 2\pounds(\beta^2) + \pounds(\beta^4), \text{ where } \beta \text{ is given in (35)}.$$

These results are not new, and were generalized by Shannon and Horadam, as well as by Brady [3]. [4]. However, their results are in terms of Lambert functions, and it is this fact which the author finds unsatisfactory, since the Lambert function is defined as an infinite series. Hence, we are using an infinite series to obtain the "closed form" sum of another infinite series; moreover, it is seen that (111) is little more than an algebraic identity, readily obtainable by manipulation of the definition in (112). It seems, therefore, that (111) is simply an artificiality, and another expression free of Lambert functions would be preferable.

It is also worth mentioning that the technique of contour integration may be used to derive identities similar to those given in Section 5. We illustrate by deriving the following identity:

(117) $$\sum_{n=1}^{\infty} \frac{\operatorname{sech} nx}{n^2 x} - \frac{\pi^2}{6x} = \frac{x}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\coth(n-\tfrac{1}{2})\pi^2/x}{(n-\tfrac{1}{2})^2}, \quad \forall x > 0.$$

Let $\underline{C}$ be the finite complex plane (z-plane), with $z = u + iv$, and consider the function $f: \underline{C} \to \underline{C}$ given by:

(118) $$f(z) = z^{-2} \operatorname{sech} xz \cot \pi z, \quad \text{where } x > \pi.$$

Let R_ξ be the residue of f at its pole ξ. Note that f is meromorphic in $\underline{C}$, with simple poles at $u_n = n$ $(n = \pm 1, \pm 2, \cdots)$ and $iv_n = (n - \tfrac{1}{2})\pi i/x$ $(n = 0, \pm 1, \pm 2, \cdots)$, and a pole of order 3 at the origin. Calculating the residues, we find:

$$R_{u_n} = \frac{\operatorname{sech} nx}{\pi n^2}; \quad R_{iv_n} = \frac{\cot(\pi i v_n)}{-v_n^2 x \sinh(x i v_n)} = \frac{(-1)^{n-1} x \coth(n-\tfrac{1}{2})y}{(n-\tfrac{1}{2})^2\pi^2},$$

where $y = \pi^2/x$; R_0 is the coefficient of z^2 in the Taylor series expansion of

$$\frac{1}{\pi}\operatorname{sech} xz \cdot \pi z \cot \pi z = \frac{1}{\pi}\left(1 - \tfrac{1}{2}(xz)^2 + \cdots\right)\left(1 - \frac{(\pi z)^2}{3} - \cdots\right);$$

hence, $R_0 = -x^2/2\pi - \pi/3$.

Now, let $\mathcal{R}_n$ be the rectangle bounded by the lines $u = \pm(N - \tfrac{1}{2})$, $v = \pm N\pi/x$, and form the sequence $(\mathcal{R}_N)_1^\infty$. It is not difficult to show that $\sec xz \cot \pi z$ is uniformly bounded on

$$\bigcup_{N=1}^{\infty} \mathcal{R}_N :$$

from this, it follows that

$$\lim_{N\to\infty} \int_{\mathcal{R}_N} f(z)dz = 0.$$

By the Cauchy Residue Theorem:

(119) $$\int_{\mathcal{R}_N} f(z)dz = \sum_{0<|u_n|<N-\frac{1}{2}} R_{u_n} + \sum_{|v_n|<N\pi/x} R_{iv_n} + R_0 .$$

Allowing N to tend to ∞ in (119), we therefore obtain:

$$\sum_{n=-\infty}^{\infty}{}' \frac{\operatorname{sech} nx}{\pi n^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}\coth (n-\tfrac{1}{2})y}{(n-\tfrac{1}{2})^2 y} = x^2/2\pi + \pi/3,$$

where the first (primed) summation excludes the term for which $n = 0$. Multiplying throughout by $\pi/2x$ and simplifying, we obtain (117). The following generalization of (117) is obtained similarly, by taking

$$f(z) = z^{-2r} \operatorname{sech} xz \cot \pi z ,$$

where r is a positive integer:

(120) $$\sum_{n=1}^{\infty} \frac{\operatorname{sech} nx}{n^{2r}x^{2r}} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{r+n}\coth (n-\tfrac{1}{2})y}{(n-\tfrac{1}{2})^{2r}y^{r-1}}$$
$$+ \sum_{k=0}^{r} \frac{(-1)^k B_{2k}E_{2r-2k}}{(2k)!(2r-2k)!} 2^{2k-1}x^{r-k}y^k = 0 ;$$

here, $y = \pi^2/x$, and the B_{2k}'s and E_{2k}'s are Bernoulli and Euler numbers, respectively.

Note that if we set $r = 0$ in (120), we obtain the apparent result:

(121) $$\sum_{n=1}^{\infty} \operatorname{sech} nx + \frac{y}{\pi}\sum_{n=1}^{\infty} (-1)^n \coth (n-\tfrac{1}{2})y + \tfrac{1}{2} = 0 .$$

By manipulations similar to those employed after (22), we may show that, for all positive y,

(122) $$\sum_{n=1}^{\infty} (-1)^{n-1}\coth (n-\tfrac{1}{2})y = \tfrac{1}{2}\sum_{n=-\infty}^{\infty} \operatorname{sech} ny .$$

Incorporating this last result into (121) and simplifying, we obtain (88), which shows that (120) is also valid for $r = 0$, though this is seemingly not justifiable by the method of contour integration. The latter method apparently provides a richer variety of identities similar to those of Section 5 than does the method of elliptic functions; on the other hand, it does not provide closed forms for the indicated series, except for special values of x and y. Thus, if we set $x = y = \pi$ in (84), (85) and (87), we obtain the results:

(123) $$\sum_{n=1}^{\infty} (-1)^{n-1} n \operatorname{csch} n\pi = \frac{1}{4\pi} ;$$

(124) $$\sum_{n=1}^{\infty} \operatorname{sech}^2 (n-\tfrac{1}{2})\pi = \frac{1}{2\pi} ;$$

$$\sum_{n=1}^{\infty} \operatorname{csch}^2 n\pi = \frac{1}{6} - \frac{1}{2\pi} . \tag{125}$$

Another important observation to make is that the identities given in this paper for real values of x and y may, with certain further restrictions, be extended to the complex plane, thereby yielding results involving corresponding trigonometric expressions, instead of hyperbolic ones. This opens up a whole new area of approach, which is beyond the scope of this paper to explore. It suffices to say that there are ample avenues of research available, as suggested in this paper, as regards the series discussed. It is hoped that sufficient interest has been generated to warrant additional investigations into the indicated topics.

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APPENDIX I – TABLE OF CONSTANTS

λ = .48121 18250 59603 ;	$1/4\lambda$ = .51952 17303 08757 ;
$\pi/4\lambda$ = 1.63212 56513 1825 ;	$\pi^2/16\lambda^2$ = 2.66383 41416 9102 ;
$y = \pi^2/2\lambda$ = 10.25494 79118 337 .	
$e^{\frac{1}{2}y}$ = 168.59071 21406 95 ;	$e^{-\frac{1}{2}y}$ = .00593 15248 58649 77 ;
e^{y} = 28,422.82822 01066 ;	e^{-y} = .00003 51829 87148 8 ;
$e^{3y/2}$ = 4,791,824.85068 042;	$e^{-3y/2}$ = .00000 02086 88762 875 ;
e^{-2y} = .00000 00012 37842 58468 ;	$e^{-5y/2}$ = .00000 00000 07342 294 ;
e^{-3y} = .00000 00000 00043 55099 975 ;	
$e^{-7y/2}$ = .00000 00000 00000 25832 ;	
e^{-4y} = .00000 00000 00000 00153 ;	$e^{-9y/2}$ = .00000 00000 00000 0000
sech $y/2$ = .01186 26323 54457 871 ;	$\operatorname{sech}^2 y/2$ = .00014 07220 46377 031;
sech y = .00007 03659 74210 458 ;	$\operatorname{sech}^2 y$ = .00000 00049 51370 327 ;
sech $3y/2$ = .00000 04173 77525 749 ;	$\operatorname{sech}^2 3y/2$ = .00000 00000 00174 204 ;
sech $2y$ = .00000 00024 75685 169 ;	$\operatorname{sech}^2 2y$ = .00000 00000 00000 006 ;
sech $5y/2$ = .00000 00000 14684 588 ;	$\operatorname{sech}^2 5y/2$ = .00000 00000 00000 000 ;
sech $3y$ = .00000 00000 00087 102 ;	

APPENDIX I – (Cont'd.)

sech $7y/2$ = .00000 00000 00000 517 ;
sech $4y$ = .00000 00000 00000 003 ;
sech $9y/2$ = .00000 00000 00000 000 .

csch $y/2$ = .01186 34671 09510 403 ;
csch $3y/2$ = .00000 04173 77525 749 ;
csch $5y/2$ = .00000 00000 14684 588 ;

$\text{csch}^2\, y/2$ = .00014 074 18 51858 435 ;
$\text{csch}^2\, 3y/2$ = .00000 00000 00174 204 ;
$\text{csch}^2\, 5y/2$ = .00000 00000 00000 000 .

$\sum_{n=-\infty}^{\infty} \text{sech}\, ny$ = 1.00014 07368 99965 ;
$\sum_{n=-\infty}^{\infty} (-1)^n \text{sech}\, ny$ = .99985 92730 02775 ;

$\sum_{n=-\infty}^{\infty} \text{sech}\, (n-\tfrac{1}{2})y$ = .02372 60994 93337 4 ;

$\sum_{n=-\infty}^{\infty} \text{sech}^2\, ny$ = 1.00000 00099 02741 ;
$\sum_{n=-\infty}^{\infty} \text{sech}^2\, (n-\tfrac{1}{2})y$ = .00028 14440 93102 469 ;

$\sum_{n=-\infty}^{\infty} \text{csch}^2\, (n-\tfrac{1}{2})y$ = .00028 14837 04065 278 .

$\sqrt{\mu}$ = .99971 85757 09592 ;
$\sqrt{1-\mu}$ = .02372 27608 25520 2 ;
$(1-\mu)^{-1/4}$ = 6.49258 11249 7349 ;
$U \equiv \tfrac{1}{2}\left\{(1-\mu)^{1/4} + (1-\mu)^{-1/4}\right\}$ = 3.32330 15370 7076 .

μ = .99943 72306 18815 ;
$(1-\mu)^{1/4}$ = .15402 19491 68033 ;

$\log(1+\sqrt{\mu})$ = .69300 64585 13859 ;
$\sqrt{5}\log(1+\sqrt{\mu})$ = 1.54960 95500 8338 ;
$\log U$ = 1.20095 87276 7835 ;

$\log(1-\sqrt{\mu})$ = -8.17564 70971 5135 ;
$\sqrt{5}\log(1-\sqrt{\mu})$ = -18.28130 26692 792 ;
$\sqrt{5}\log U$ = 2.68542 53532 6045 .

ρ = 1.63235 53516 2277 ;
$\rho\sqrt{5}$ = 3.65005 75296 6408 ;
$\rho\sqrt{1-\mu}$ = .03872 39755 88805 0 ;
$\rho(1-\mu)^{1/4}$ = .25141 85529 91809 .

$\rho\sqrt{\mu}$ = 1.63189 59671 7624 ;
$\rho\sqrt{5\mu}$ = 3.64903 03148 1385 ;
$\rho\sqrt{5(1-\mu)}$ = .08658 94417 76510 3 ;

ρ^2 = 2.66458 39939 7149 ;
$\rho^2\sqrt{5}$ = 5.95819 09422 7815 ;
$\rho^2\sqrt{1-\mu}$ = .06321 12887 88495 2 ;
$\rho\sigma$ = .52027 15562 09976 ;

$\rho^2\mu$ = 2.66308 44476 8609 ;
$\rho^2\mu\sqrt{5}$ = 5.95483 78548 4858 ;
$\rho^2\sqrt{5(1-\mu)}$ = .14134 47386 76446 ;
$\rho\sigma\sqrt{5}$ = 1.16336 25664 4511 .

APPENDIX II – COMPUTED FORMULAS FOR SERIES IN SECTION A

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n-1}} = .81594\ 79835\ 88122\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = 1.82451\ 51574\ 0692\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = .56617\ 76758\ 11385\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{2n}} = .23063\ 80122\ 05598\ ;$$

$$\sum_{n=1}^{\infty} \frac{2n}{F_{2n}} = 4.79482\ 83758\ 3304\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2n}{F_{2n}} = 1.16000\ 94790\ 1554\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = .13513\ 57781\ 04988\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2} = 1.12939\ 07263\ 5581\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = 1.07215\ 62188\ 8076\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} = 1.29693\ 00248\ 1143\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)F_{4n-2}} = 1.04573\ 08199\ 4974\ ;$$

$$\sum_{n=1}^{\infty} \frac{4n-2}{F_{4n-2}} = 2.97741\ 89274\ 2429\ ;$$

$$\sum_{n=1}^{\infty} \frac{4n}{F_{4n}} = 1.81740\ 94484\ 0875\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}} = .39840\ 78440\ 08491\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{4n}} = .16776\ 98318\ 02894\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = 1.20729\ 19969\ 8575\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{n}^2} = 2.42632\ 07511\ 6724\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{L_{4n}} = .12429\ 07235\ 04095\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{4n-2}} = .89086\ 70219\ 72118\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{4n}^2} = .02087\ 07112\ 49618\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2} = .11342\ 79589\ 57717\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{4n-2}^2} = .11426\ 50668\ 55370\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^2} = 1.01596\ 27673\ 9809\ ;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4n}{F_{4n}} = 1.02201\ 78277\ 6866\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)L_{2n-1}} = 1.10858\ 16944\ 5815\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)F_{8n-4}} = .33567\ 81691\ 57557\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{8n}} = .02173\ 95541\ 49399\ ;$$

$$\sum_{n=1}^{\infty} \frac{1}{L_{8n-4}} = .14603\ 02776\ 53494\ ;$$

$$\sum_{n=1}^{\infty} \frac{8n}{F_{8n}} = .39769\ 58103\ 20044\ ;$$

$$\sum_{n=1}^{\infty} \frac{8n-4}{F_{8n-4}} = 1.41971\ 36380\ 8871\ .$$

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