A NOTE ON THE SUMMATION OF SQUARES

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Consider

$$P_{n+2} = pP_{n+1} + qP_n, \quad P_0 = 0, \quad P_1 = 1.$$

We wish to find

(A) $$\sum_{j=1}^{n} P_j^2 = P_n P_{n+1} \quad \text{if } p = q = 1;$$

(B) $$\sum_{j=1}^{n} P_j^2 = \frac{P_n P_{n+1}}{p} \quad \text{if } q = 1;$$

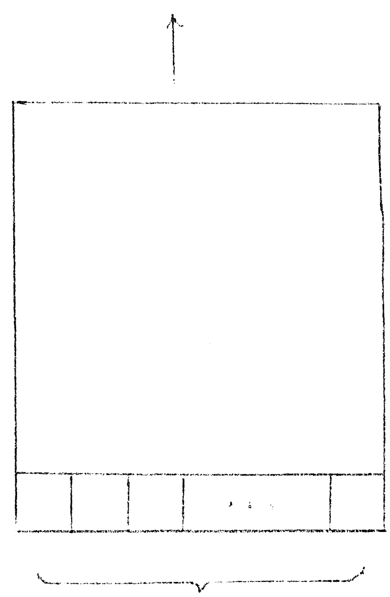
(C) $$\sum_{j=1}^{n} P_j^2 = \frac{2q^2 P_{n+1} P_n + \frac{(1-q)}{p}\,[P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]}{q(p^2+q^2) - (p-q)^2} .$$

The usual way to establish (A) is by induction after (A) has been guessed from tabular data, or by the geometric method of Brother Alfred [1]. We now establish (B) by the method of [1].

Form p unit squares horizontally. Above these add p copies of $p \times p$ squares. This yields

$$p \cdot (p^2 + 1) = P_2 P_3 .$$

Add to the left p copies of the square P_2 on the edge to get a rectangle $P_3 P_4$.



p unit squares

Since every square P_1, P_2, P_3 is used p times so far

$$P_1^2 + P_2^2 + P_3^2 = P_3P_4/p.$$

This obviously may be continued as far as one wishes so that

$$\sum_{j=1}^{n} P_j^2 = P_nP_{n+1}/p, \quad p \neq 0, \quad q = 1.$$

Second Method: $(q = 1)$

Start with

$$P_{n+2} = pP_{n+1} + P_n$$

and multiply through by P_{n+1} to get

$$P_{n+1}P_{n+2} = pP_{n+1}^2 + P_nP_{n+1}$$

$$\sum_{j=0}^{n} P_{j+2}P_{j+1} = \sum_{j=0}^{n} pP_{j+1}^2 + \sum_{j=0}^{n} P_jP_{j+1}.$$

Thus,

$$P_{n+2}P_{n+1} = p\sum_{j=0}^{n} P_{j+1}^2 = p\sum_{j=1}^{n+1} P_j^2 \quad \text{and} \quad \sum_{j=1}^{n} P_j^2 = P_nP_{n+1}/p.$$

Before doing the general case, let us consider the result $p = 1$ and $q \neq 0$.

$$P_{n+2} = P_{n+1} + qP_n$$

$$P_{n+2}P_{n+1} = P_{n+1}^2 + qP_{n+1}P_n$$

$$qP_{n+1}P_n = qP_n^2 + q^2P_nP_{n-1}$$

$$q^2P_nP_{n-1} = q^2P_{n-1}^2 + q^3P_{n-1}P_{n-2}$$

$$\cdots \qquad \cdots$$

$$q^{n-1}P_2P_1 = q^{n-1}P_1^2 + q^nP_1P_0.$$

Thus,

$$\sum_{j=0}^{n} q^jP_{n+1-j}^2 = P_{n+1}P_{n+2}.$$

We now proceed to the general case. From

$$P_{n+2}P_{n+1} = pP_{n+1}^2 + qP_nP_{n+1}$$

one may at once write

$$\text{(D)} \qquad \sum_{j=1}^{n+1} pP_j^2 = P_{n+2}P_{n+1} + (1-q)\sum_{j=1}^{n} P_jP_{j+1},$$

while from

$$P_{j+2}^2 = p^2P_{j+1}^2 + q^2P_j^2 + 2pqP_jP_{j+1}$$

one can immediately write

$$\text{(E)} \qquad P_{n+2}^2 + P_{n+1}^2 - P_2^2 - P_1^2 = p^2(P_{n+1}^2 - P_1^2) + (p^2+q^2-1)\sum_{j=1}^{n} P_j^2 + 2pq\sum_{j=1}^{n} P_jP_{j+1}.$$

One can now use (D) and (E) to solve directly for

$$\sum_{j=1}^{n+1} pP_j^2 = P_{n+2}P_{n+1} + (1-q)\sum_{j=1}^{n} P_jP_{j+1} = P_{n+2}P_{n+1} + \frac{(1-q)}{2pq}\left\{P_{n+2}^2 + P_{n+1}^2 - p^2 - 1 - p^2P_{n+1}^2 + p^2\right.$$

$$\left. = (p^2+q^2-1)\sum_{j=1}^{n} P_j^2\right\}$$

$$pP_{n+1}^2 + \left(\sum_{j=1}^{n} P_j^2\right)\left(p - \frac{(1-q)(p^2+q^2-1)}{2pq}\right) = P_{n+2}P_{n+1} + \frac{1-q}{2pq}\,[P_{n+2}^2 + P_{n+1}^2 - 1 - p^2P_{n+1}^2]$$

$$\sum_{j=1}^{n} pP_j^2 = \frac{P_{n+2}P_{n+1} - pP_{n+1}^2 + \dfrac{(1-q)}{2pq}\,[P_{n+2}^2 + P_{n+1}^2(1-p^2) - 1]}{(2pq - p^2 - q^2 + 1 + qp^2 + q^3 - 1)/2pq}$$

Testing $p = 1,\ q = 1,$

$$\sum_{i=1}^{n} F_i^2 = \frac{2F_{n+2}F_{n+1} - 2F_{n+1}^2}{2} = F_{n+1}F_n\ .$$

For $q = 1$ only,

$$\sum_{i=1}^{n} pP_i^2 = \frac{2pP_{n+2}P_{n+1} - 2p^2P_{n+1}^2}{p^2 + 1 - (p-1)^2} = \frac{P_{n+2}P_{n+1} - pP_{n+1}^2}{2p} = P_{n+1}P_n$$

so that

$$\sum_{i=1}^{n} P_i^2 = P_{n+1}P_n/p\,.$$

Thus,

$$\sum_{j=1}^{n} P_j^2 = \frac{2qP_{n+2}P_{n+1} - 2pqP_{n+1}^2 + \dfrac{(1-q)}{p}\,[P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]}{q(p^2+q^2) - (p-q)^2}$$

$$= \frac{2q^2(P_{n+1}P_n) + \dfrac{(1-q)}{p}\,[P_{n+2}^2 + (1-p^2)P_{n+1}^2 - 1]}{q(p^2+q^2) - (p-q)^2}\ .$$

REFERENCE

1. Brother Alfred Brousseau, "Fibonacci Numbers and Geometry," *The Fibonacci Quarterly*, Vol. 10, No. 3 (April, 1972), pp. 303–318⁺.

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