FIBONACCI SEQUENCES AND ADDITIVE TRIANGLES OF HIGHER ORDER AND DEGREE

THOMAS B. KIRKPATRICK, JR.
Chicago, Illinois

It is often desirable for either ease of calculation or nicety of expression to represent a function in terms of positive integers only. For example, the Binet formula for the N^{th} term of the Fibonacci series quite easily reduces to the expansion:

$$F_N = \frac{\binom{N}{1}5^0 + \binom{N}{3}5^1 + \binom{N}{5}5^2 + \binom{N}{7}5^3 + \cdots}{2^{N-1}} \quad .$$

The last term, of course, will be $5^{(N-1)/2}$ if N is odd, or $N5^{(N-2)/2}$ if N is even.

However, it is well known that the sums of the terms of the ascending diagonals of the Pascal triangle also produce the Fibonacci numbers, thus providing another simple expansion,

$$F_N = \binom{N-1}{0} + \binom{N-2}{1} + \binom{N-3}{2} + \binom{N-4}{3} + \cdots,$$

the last term being $N/2$ if N is even and 1 if N is odd. (This comes as no surprise since a common method of constructing the triangle is by a two-step additive process.)

$F_N = 0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55$

N											$\sum = 2^N =$
$N = 0$	1										1
1	1	1									2
2	1	2	1								4
3	1	3	3	1							8
4	1	4	6	4	1						16
5	1	5	10	10	5	1					32
6	1	6	15	20	15	6	1				64
7	1	7	21	35	35	21	7	1			128
8	1	8	28	56	70	56	28	8	1		256
9	1	9	36	84	126	126	84	36	9	1	512
10											

It is interesting to note that since the sum of the terms of the N^{th} diagonal is equal to F_N, and the sum of the terms of the N^{th} row is equal to 2^N, then the product of those two sums is equal to twice the numerator of the first expansion:

$$2^N\left[\binom{N}{1}5^0 + \binom{N}{3}5^1 + \binom{N}{5}5^2 + \binom{N}{7}5^3 + \cdots\right]$$

or

$$\frac{(1+\sqrt{5})^N - (1-\sqrt{5})^N}{\sqrt{5}} \quad .$$

A Tribonacci triangle, constructed by a three-step additive process, has as the sum of the terms of the N^{th} row 3^N, provides the coefficients of the expansion $(x^0 + x^1 + x^2)^N$, and has the Tribonacci numbers as sums of the ascending diagonals:

$T_N =$	0	1	1	2	4	7	13	24	44							$\sum = 3^N =$
$N = 0$	1															1
1	1	1	1													3
2	1	2	3	2	1											9
3	1	3	6	7	6	3	1									27
4	1	4	10	16	19	16	10	4	1							81
5	1	5	15	30	45	51	45	30	15	5	1					243
6	1	6	21	50	90	126	141	126	90	50	21	6	1			729
7	1	7	28	77	161	266	357	393	357	266	161	77	28	7	1	2187
8	1	8	36	...												

Fig. 2 Tribonacci Triangle

Just as the terms in each row of the Pascal triangle are the binomial coefficients, the terms of each row of the Tribonacci triangle are the trinomial coefficients; that is, if the trinomial expression $(x^0 + x^1 + x^2)$ is raised to a given power such as three,

$$(x^0 + 3x^1 + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6),$$

the coefficients of the resulting terms are the terms of the corresponding row $(N = 3)$ of the Tribonacci triangle, 1, 3, 6, 7, 6, 3, 1.

An easy method of constructing the triangle, rather than actually multiplying the trinomials or using a generating formula for each term (which is simple for Pascal's triangle, but much more complex for higher order triangles), is to simply create each term by adding the three terms immediately above and to the left in the preceding row. For example, the fourth row is derived from the third row as follows:

$N = 3$ 1 3 6 7 6 3 1 $\sum = 3^3$

4 1 4 10 16 19 16 10 4 1 3^4

(0 + 0 + 1 = 1)

(0 + 1 + 3 = 4)

(1 + 3 + 6 = 10)

(3 + 6 + 7 = 16)

(6 + 7 + 6 = 19)

(7 + 6 + 3 = 16)

(6 + 3 + 1 = 10)

(3 + 1 + 0 = 4)

(1 + 0 + 0 = 1)

Any additive triangle must begin with the number 1, since any quantity with an exponent of zero is by definition 1. The second row is also composed of ones, since the coefficients of the given trinomial are 1, 1 and 1. From this point onward all terms can easily be calculated by the process described above, which is in effect an arithmetical short-cut through the lengthy process of multiplying polynomials of ascending powers of X.

The ascending diagonals of a four-step or Quadronacci triangle provide the terms of that series in a similar manner:

$Q_N = 0 \quad 1 \quad 1 \quad 2 \quad 4 \quad 8 \quad 15 \quad 29$

N =														$\sum = 4^N =$
0	1													1
1	1	1	1	1										4
2	1	2	3	4	3	2	1							16
3	1	3	6	10	12	12	10	6	3	1				64
4	1	4	10	20	31	40	44	40	31	20	10	4	1	256
5	1	5	...											
6	1	...												
7														

Fig. 3. Quadranacci Triangle $(x^0 + x^1 + x^2 + x^3)^N$

The terms in the Quadronacci triangle are derived just as are those in the Tribonacci triangle, except that the terms are added in groups of four instead of three.

Indeed, any order of additive triangle can be thus constructed to give similar results. If the order, K, of an additive triangle is defined to be the number of terms in its polynomial base, which is also the number of additive steps necessary to derive each term from the terms of the preceding row, then it may be noted that for an additive triangle of K^{th} order: each row will have $(K - 1)$ more terms than the preceding row; the sum of the terms of each row will equal K^N; and the terms in each row will provide the coefficients of the expansion:

$$(x^0 + x^1 + x^2 + x^3 + \cdots + x^{K-1})^N.$$

The sums of the terms of the ascending diagonals of a K^{th} order triangle produce the series:

$$T_N + T_{N+1} + T_{N+2} + T_{N+3} + \cdots + T_{N+K-1} = T_{N+K}$$

$$T_1 = 1, \quad T_2 = 1; \quad (K-1) \geqslant 1.$$

Since for Pascal's triangle $K = 2$ (see Fig. 1), the series produced is $T_N + T_{N+1} = T_{N+2}$, the Fibonacci series. Similarly, in the Quadranacci triangle, since $K = 4$, the series produced is

$$T_N + T_{N+1} + T_{N+2} + T_{N+3} = T_{N+4}, \quad T_3 = 2, \quad T_4 = 4.$$

If an additive triangle is altered by arranging the ascending diagonals as rows, a corresponding alteration results in the series produced by the sums of the terms of the new ascending diagonals. The new series consists of the same number of steps, but of different terms. For example, the Pascal triangle (see Fig. 1), when altered in this manner, now becomes:

$F'_N = 0 \quad 1 \quad 1 \quad 1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 9 \quad 13 \quad 19$

N =						$\sum = F_{N+1} =$
0	1					1
1	1					1
2	1	1				2
3	1	2				3
4	1	3	1			5
5	1	4	3			8
6	1	5	6	1		13
7	1	6	10	4		21
8	1	7	15	10	1	34
9	1	8	21	20	5	55
10	1	...				

Fig. 4 Second-Degree Pascal Triangle

The ascending diagonals provide the terms of the series

$$T_N + T_{N+2} = T_{N+3} \qquad (T_1 = T_2 = T_3 = 1).$$

A similar treatment of the Tribonacci triangle produces the series

$$T_N + T_{N+2} + T_{N+4} = T_{N+5} \qquad (T_1 = T_2 = T_3 = 1).$$

It can readily be seen that these two new additive series skip every other term.

If the new ascending diagonals are converted into rows a second time for the Tribonacci triangle, the sums of the terms of the resulting diagonals will produce the series

$$T_N + T_{N+3} + T_{N+6} = T_{N+7} \qquad (T_1 = 1, T_2 = 1, T_3 = 1, T_4 = 1).$$

Here the series skips twice between each term.

If the degree, R, of an additive triangle is defined to be the number of times the triangle has been altered by rearranging the ascending diagonals into rows (beginning with $R = 1$ for the triangle in unaltered form), it may then be said that for an additive triangle of K^{th} order and R^{th} degree, the sums of the terms of the ascending diagonals produce the series:

$$T_N + T_{N+R} + T_{N+2R} + T_{N+3R} + \cdots + T_{N+R(K-1)} = T_{N+R(K-1)+1}$$

$$(T_1 = 1,\ T_2 = 1,\ T_3 = 1, \cdots, T_{R+1} = 1) \qquad (K-1) \geqslant 1,\ R \geqslant 1.$$

For the standard Pascal triangle, since $K = 2$ and $R = 1$, the series is the normal Fibonacci series (1, 1, 2, 3, 5, 8, ⋯), where

$$T_N + T_{N+1} = T_{N+2} \qquad (T_1 = T_2 = 1).$$

For a five-step additive triangle, the diagonals of which have been twice rearranged into rows $(K = 5,\ R = 3)$, the series produced is

1, 1, 1, 1, 2, 3, 4, 6, 8, 13, 19, 28, 41, 60, 88, 129, 188, ⋯ ,

where

$$T_N + T_{N+3} + T_{N+6} + T_{N+9} + T_{N+12} = T_{N+13} \qquad (T_1 = T_2 = T_3 = T_4 = 1).$$

Comparing Fig. 1 with Fig. 4, it will be observed that the terms in each column remained unaltered by a change in the degree of the triangle; each column is merely lowered with respect to the column to its left. Consequently, if the terms of the N^{th} row (and hence the terms of the columns) of a first degree K^{th} order triangle can be expressed in terms of N, then it follows that the N^{th} term of the additive series produced by the sums of the terms of the ascending diagonals of a K^{th} order R^{th} degree triangle can be expressed as a seriers in N and R. For example, the sums of the terms of the ascending diagonals of the Pascal triangle $(K = 2)$ of R^{th} degree produce the series:

$$T_N + T_{N+R} = T_{N+R+1}.$$

The N^{th} term of this series is the expansion

$$T_N = \binom{N-1}{0} + \binom{N-1-R}{1} + \binom{N-1-2R}{2} + \binom{N-1-3R}{3} + \cdots$$

It is easy to conjecture that a general expansion in terms of N, K and R is possible for the N^{th} term of the series generated by the sums of the terms of the ascending diagonals of a triangle of K^{th} order and R^{th} degree, but that requires a treatment much more advanced than is offered here.

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