CERTAIN GENERAL BINOMIAL-FIBONACCI SUMS

J. W. LAYMAN
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

Numerous writers appear to have been fascinated by the many interesting summation identitites involving the Fibonacci and related Lucas numbers. Various types of formulas are discussed and various methods are used. Some involve binomial coefficients [2], [4]. Generating function methods are used in [2] and [5] and higher powers appear in [6]. Combinations of these or other approaches appear in [1], [3] and [7].

One of the most tantalizing displays of such formulas was the following group of binomial-Fibonacci identities given by Hoggatt [5]. He gives:

$$1^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k, \tag{1}$$

$$2^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{3k}, \tag{2}$$

$$3^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{4k}. \tag{3}$$

In these formulas and throughout this paper F_n denotes the n^{th} Fibonacci number defined by the recurrence:

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, \qquad F_1 = 1. \tag{4}$$

Hoggatt attributes formula (2) to D. A. Lind, (3) to a special case of Problem 3-88 in the Fibonacci Quarterly and states that (1) is well known.

The three identities given above suggest, rather strongly, the possibility of a general formula of which those given are special instances. Hoggatt does obtain many new sums but does not appear to have succeeded in obtaining a satisfactory generalization of formulas (1)—(3).

In the present paper, we give elementary, yet rather powerful, methods which yield many general binomial-Fibonacci summation identities. In particular, we obtain a sequence of sums the three simplest members of which are precisely the formulas (1)—(3) given above. In addition, similar families of sums are obtained with the closed forms $a^n_{1,m} F_n$ and $a^n_{3,m} F_{3n}$ for $m = 1, 2, 3, \cdots$, as well as the general two-parameter family of sums with the closed form $(a_{r,m})^n F_{rn}$.

Our principal tools for obtaining sums will be the binomial expansion formula

$$\sum_{k=0}^{m} \binom{m}{k} (y-1)^k = y^m, \tag{5}$$

and the fact that the Fibonacci number F_n is a linear combination of a^n and b^n, where

$$a = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad b = \frac{1-\sqrt{5}}{2}$$

are roots of the polynomial equation

$$x^2 = x + 1. \tag{6}$$

The Fibonacci numbers are then

$$F_n = (1/\sqrt{5})(a^n - b^n). \tag{7}$$

We are already in a position to obtain a summation formula. Let w stand for a root a or b of (6). Then we have

(8) $$w^2 = w + 1.$$

Clearly, then, by (5) and (8),

$$(w^2)^n = \sum_{u=0}^{n} \binom{n}{k} (w^2 - 1)^k = \sum_{n=0}^{n} \binom{n}{k} w^k,$$

and therefore

$$(a^2)^n - (b^2)^n = \sum_{n=0}^{n} \binom{n}{k} (a^k - b^k).$$

But from (7), this is seen to be equivalent to

$$F_{2n} = \sum_{n=0}^{n} \binom{n}{k} F_k,$$

which is formula (1).

In order to obtain more general results, we proceed as follows. From (8) we see that

$$w^2 = w + 1 = F_2 w + F_1.$$
$$w^3 = w^2 + w = F_3 w + F_2,$$

and, in general, by an easy induction,

(9) $$w^m = w^{m-1} + w^{m-2} = F_m w + F_{m-1}.$$

Rewriting, we have

$$1 - \frac{w^m}{F_{m-1}} = -\frac{F_m}{F_{m-1}} w, \quad m \neq 1,$$

or, equivalently,

(10) $$-\left(\frac{F_m}{F_{m-1}}\right)^n w^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-1}{F_{m-1}}\right)^k w^{mk},$$

where, again, w may be either a or b. Again using the fact that F_n is a linear combination of a^n and b^n, we obtain

(11) $$\left(\frac{F_m}{F_{m-1}}\right)^n F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} \left(\frac{1}{F_{m-1}}\right)^k F_{mk}, \quad m \neq 1.$$

Equation (11) takes on especially simple forms for certain values of m. For example, when $m = 2$ and 3, respectively, we have

(12) $$F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} F_{2k}$$

and

(13) $$2^n F_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+n} F_{3k}.$$

Other values of m result in non-integral ratios in (11), e.g., $m = 4$ and 5 give

(14) $$\left(\frac{3}{2}\right)^n F_n = (-1)^n \sum_{k=0}^{n} \binom{h}{k} (-\tfrac{1}{2})^k F_{4k}$$

and

(15) $$\left(\frac{5}{3}\right)^n F_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1/3)^k F_{5k},$$

Each of the sums (12)–(15) and the general sum in (11) yield closed forms of the type

$$(a_{1,m})^n F_n .$$

In order to obtain sums with closed forms of the type

$$(a_{2,m})^n F_{2n}$$

we return to (9). If we let $m = 2$ and solve for w, $w = w^2 - 1$, we may substitute this expression into (9) to obtain

(16) $$w^m = F_m(w^2 - 1) + F_{m-1} = F_m w^2 - (F_m - F_{m-1}) = F_m w^2 - F_{m-2} .$$

This is equivalent to:

(17) $$\frac{F_m}{F_{m-2}} w^2 = \frac{1}{F_{m-2}} w^m + 1, \quad m \neq 2.$$

Now proceeding in the same manner as led to (11) results in the general formula

(18) $$(F_m)^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} (F_{m-2})^{n-k} F_{mk}, \quad m \neq 2 .$$

The special cases $m = 1, 3,$ and 4 of this general equation are found to give exactly the three sums involving F_{2n} which were listed by Hoggatt and given above in (1)–(3). All other cases can easily be seen to lead to formulas containing a power of a Fibonacci number in the summand and in this sense previous investigators can be said to have found all "easy" sums of this type. The first two cases giving new sums are thus, for $m = 5$ and 6,

(19) $$5^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_{5k}$$

and

(20) $$8^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{6k} .$$

Steps similar to those leading to (16) can be followed to express w^m in terms of w^3. We find, after simplifying,

$$2w^m = F_m w^3 + F_{m-3}$$

which, following our general procedure, yields

(21) $$(F_m)^n F_{3n} = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} 2^k (F_{m-3})^{n-k} F_{mk}, \quad m \neq 3 .$$

For $m = 2, 4, 5, 6$ we have, respectively,

(22) $$F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{2k},$$

(23) $$3^n F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{4k},$$

(24) $$5^n F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{5k} ,$$

and

$$4^n F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{6k}. \tag{25}$$

Rather than continuing with these special families of sums, we now proceed to the general two-parameter family yielding closed forms of the type

$$(a_{r,m})^n F_{rn}.$$

Let $0 < r < m$. From (9) we have

$$w^r = F_r w + F_{r-1}, \qquad w^m = F_m w + F_{m-1}$$

which give, after considerable simplification,

$$w^m = \frac{F_m}{F_r} w^r + (-1)^{r-1} \frac{F_{m-r}}{F_r}, \qquad 0 < r < m. \tag{26}$$

The result just obtained is equivalent to

$$(-1)^r \left(\frac{F_m}{F_{m-r}} \right) w^r = (-1)^r \left(\frac{F_r}{F_{m-r}} \right) w^m + 1, \qquad 0 < r < m, \tag{27}$$

which yields the summation

$$(F_m)^n F_m = \sum_{k=0}^{n} \binom{n}{k} (-1)^{r(n-k)} (F_{m-r})^{n-k} (F_r)^k F_{mk}, \tag{28}$$

valid for all integral $m,n,$ and r satisfying $0 < r < m$.

A number of special cases of the above general formula have been given previously in this paper for $r = 1, 2,$ and 3. Another interesting case results when $m = 2r$. Using the well known fact that F_{2r}/F_r is the Lucas number L_r defined by the recurrence

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2, \qquad L_1 = 1, \tag{29}$$

we have, in this case,

$$(L_r)^n F_{rn} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{r(n-k)} F_{2rk}. \tag{30}$$

The special case $r = 2p$ has been obtained by Hoggatt in [5]. Some instances of (30) which have not been given among our previous formulas are

$$7^n F_{4n} = \sum_{k=0}^{n} \binom{n}{k} F_{8k} \tag{31}$$

and

$$11^n F_{5n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_{10k} \tag{32}$$

which obtain when $r = 4$ and 5, respectively.

Of course, if we recall that the Lucas numbers L_n are linear combinations of a^n and b^n, defined in (7), specifically

$$L_n = a^n + b^n, \tag{33}$$

then we see that each sum obtained above remains valid when L_n is substituted for F_n at the appropriate occurrences of F_n in each formula. We state some of these. From (18) we have

(34) $$(F_m)^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} (F_{m-2})^{n-k} L_{mk}, \qquad m \neq 2,$$

several specific instances being

$$1^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_k,$$

$$2^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_{3k}$$

and

$$3^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_{4k}.$$

The interested reader may obtain other Lucas number analogs of formulas given above.

Preliminary results indicate that modifications of the methods used in this paper will lead to many other quite general results on binomial Fibonacci sums. Perhaps we might be forgiven for paraphrasing Professor Moriarty (see [4]) in saying "many beautiful results have been obtained, many yet remain."

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