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SOME CLASSES OF FIBONACCI SUMS

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1. INTRODUCTION

Layman [3] recalled the formulas [2]

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k, \tag{1.1}$$

$$2^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{3k}, \tag{1.2}$$

$$3^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_{4k}, \tag{1.3}$$

where, as usual, the F_n are the Fibonacci numbers defined by

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1).$$

As Layman remarks, the three identities suggest the possibility of a general formula of which these are special instances. Several new sums are given in [2]. Many additional sums occur in [1].

Layman does not obtain a satisfactory generalization; however, he does obtain a sequence of sums that include (1.1), (1.2), and (1.3). In particular, the following elegant formulas are proved:

$$5^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_{5k}, \tag{1.4}$$

$$8^n F_{2n} = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{6k}, \tag{1.5}$$

$$F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{2k}, \tag{1.6}$$

$$5^n F_{3n} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-2)^k F_{5k}. \tag{1.7}$$

He notes also that each of the sums he obtains remains valid when F_n is replaced by L_n, where the L_n are the Lucas numbers defined by

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1} \qquad (n \geq 1).$$

In the present paper, we consider the following question. Let p, q be fixed positive integers. We seek all pairs λ, μ such that

$$\lambda^n F_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk} \qquad (n = 0, 1, 2, \ldots). \tag{1.8}$$

It is easily seen that $p \neq q$. We shall show that (1.8) holds if and only if

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}. \tag{1.9}$$

Since (1.8) is equivalent to

$$(-\mu)^n F_{qn} = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^k F_{pk} \qquad (n = 0, 1, 2, \ldots), \tag{1.10}$$

we may assume that $p < q$. However, this is not necessary since we may take $F_{-n} = (-1)^{n-1}F_n$. Also, the final result is in fact for all p, q, $p \neq q$.

For the Lucas numbers, we consider

$$\lambda^n L_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k L_{qk} \qquad (n = 0, 1, 2, \ldots). \tag{1.11}$$

We show that (1.11) holds if and only if λ, μ satisfy (1.9) or

$$\lambda = \frac{F_q}{F_{p+q}}, \quad \mu = -\frac{F_p}{F_{p+q}}. \tag{1.9$'$}$$

In the next place, if w denotes a root of $x^2 = x + 1$, we show that

$$\lambda^n w^{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k w^{qk} \qquad (n = 0, 1, 2, \ldots), \tag{1.12}$$

if and only if λ, μ satisfy (1.9).

The stated results concerning (1.8) and (1.11) can be carried over to the more general

$$\lambda^n F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk+r} \qquad (n = 0, 1, 2, \ldots) \tag{1.13}$$

and

$$\lambda^n L_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k L_{qk+r} \qquad (n = 0, 1, 2, \ldots), \tag{1.14}$$

where r is an arbitrary integer. We show that (1.13) holds if and only if λ, μ satisfy (1.9); thus, the result for (1.13) includes that for (1.8). However, (1.14), with $r \neq 0$, holds if and only if λ, μ satisfy (1.9); thus, the result for (1.13) includes that for (1.8). But (1.14), with $r \neq 0$, holds if and only if λ, μ satisfy (1.9); thus, the values (1.9)$'$ for λ, μ apply only in the case $r = 0$.

As for

$$\lambda^n w^{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k w^{qk+r} \qquad (n = 0, 1, 2, \ldots),$$

it is obvious that this is equivalent to (1.12) for all r.

The formulas (1.8), (1.11), (1.12), (1.13), (1.14) with λ, μ satisfying (1.9) can all be written in such a way that they hold for all p, q. For example, (1.8) becomes

$$F_q^n F_{pn} = \sum_{k=0}^{n} (-1)^{p(n-k)} \binom{n}{k} F_p F_{q-p}^{n-k} F_{qk}. \tag{1.15}$$

For $p = q$, this reduces to a mere tautology. However, for (1.11) with λ, μ defined by (1.9), we have

$$F_q^n L_{pn} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_p F_{p+q}^{n-k} L_{qk}. \tag{1.16}$$

For $q = p$, this reduces to

$$L_{pn} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_p^{n-k} L_{pk}. \tag{1.17}$$

Note that (1.15) and (1.16) had been obtained in [1].

For some remarks concerning (1.17) see §7 below. In particular, the following pair of formulas is obtained:

$$(-1)^r L_{pn-r} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_p^{n-k} L_{pk+r}, \tag{1.18}$$

$$(-1)^{r-1} F_{pn-r} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_p^{n-k} F_{pk+r}, \tag{1.19}$$

where r is an arbitrary integer.

Formulas (1.18) and (1.19) differ from (1.13) and (1.14) in a rather essential way. The former pair suggest the problem of determining λ, μ, C_r such that

$$C_r \lambda^n L_{pn-r} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mu^k L_{pk+r},$$

and similarly for

$$C_r \lambda^n F_{pn-r} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mu^k F_{pk+r},$$

where C_r depends only on r. This is left for another paper.

SECTION 2

Let a, b denote the roots of $x^2 = x + 1$. We recall that

$$F_n = \frac{a^n - b^n}{a - b}, \quad L_n = a^n + b^n. \tag{2.1}$$

Thus, the equation

$$\lambda^n F_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk} \tag{2.2}$$

becomes

$$\lambda^n(a^{pn} - b^{pn}) = \sum_{k=0}^{n} \binom{n}{k} \mu^k (a^{qk} - b^{qk}). \tag{2.3}$$

Multiplying both sides of (2.3) by x and summing over n we get

$$\begin{aligned}
\frac{1}{1 - \lambda a^p x} - \frac{1}{1 - \lambda b^p n} &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} \mu^k (a^{qk} - b^{qk}) \\
&= \sum_{k=0}^{\infty} \mu^k (a^{qk} - b^{qk}) x^k \sum_{n=0}^{\infty} \binom{n+k}{k} x^n \\
&= \sum_{k=0}^{\infty} \mu^k (a^{qk} - b^{qk}) \frac{x^k}{(1 - x)^{k+1}} \\
&= \frac{1}{1 - x} \left\{ \frac{1}{1 - \dfrac{\mu a^q x}{1 - x}} - \frac{1}{1 - \dfrac{\mu b^q x}{1 - x}} \right\}.
\end{aligned}$$

Since

$$\frac{1}{a - b}\left(\frac{1}{1 - a^p z} - \frac{1}{1 - b^p z}\right) = \frac{1}{1 - L_p z + (-1)^p z^2},$$

it follows that

$$\frac{\lambda F_p}{1 - \lambda L_p x + (-1)^p \lambda^2 x^2} = \frac{\mu F_q}{(1 - x)^2 - \mu L_q x(1 - x) + (-1)^q \mu^2 x^2}. \tag{2.4}$$

For $x = 0$, this reduces to

$$\lambda F_p = \mu F_q. \tag{2.5}$$

Thus,

$$1 - \lambda L_p x + (-1)^p \lambda^2 x^2 = (1 - x)^2 - \mu L_q x(1 - x) + (-1)^q \mu^2 x^2. \tag{2.6}$$

Equating coefficients of x and x^2, we get

$$\lambda L_p = 2 + \mu L_q \tag{2.7}$$

and

$$(-1)^p \lambda^2 = 1 + \mu L_q + (-1)^q \mu^2, \tag{2.8}$$

respectively.

Now by (2.5) and (2.7), we have

$$\lambda L_p F_q = 2F_q + \mu L_p L_q = 2F_q + \lambda F_p L_q,$$

so that

$$\lambda(L_p F_q - F_p L_q) = 2F_q. \tag{2.9}$$

It is easily verified that

$$L_p F_q - F_p L_q = 2(-1)^{q-1} F_{p-q}. \tag{2.10}$$

Hence, (2.9) yields

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}; \tag{2.11}$$

the second equality is of course a consequence of (2.5).

It remains to consider the condition (2.8). We shall show that (2.8) is implied by (2.11), or, what is the same, by (2.5) and (2.7). To do this with a minimum of computation, note that (2.5), (2.7), (2.8) can be replaced by

$$(2.5)' \qquad \lambda(a^p - b^p) = (1 + \mu a^q) - (1 + \mu b^q),$$

$$(2.7)' \qquad \lambda(a^p + b^p) = (1 + \mu a^q) + (1 + \mu b^q),$$

$$(2.8)' \qquad \lambda^2(ab)^p = (1 + \mu a^q)(1 + \mu b^q),$$

respectively. Subtracting the square of (2.5)′ from the square of (2.7)′, we get (2.8)′.

We have therefore proved that (2.5) and (2.7) imply both (2.8) and (2.11). Conversely, (2.11) implies (2.5) and (2.7). The first implication, (2.11) (2.5) is immediate. As for (2.11) → (2.7), we have

$$\lambda L_p - \mu L_q = (-1)^p \cdot \frac{L_pF_q - F_pL_q}{F_{q-p}} = (-1)^q \cdot \frac{2(-1)^pF_{q-p}}{F_{q-p}},$$

by (2.10). Hence, $\lambda L_p - \mu L_q = 2$.

This completes the proof of the following:

Theorem 1: Let p, q be fixed positive integers, $p \neq q$. Then,

$$(2.12) \qquad \lambda^n F_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk} \qquad (n = 0, 1, 2, \ldots),$$

if and only if

$$(2.13) \qquad \lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}.$$

Thus, we have the explicit identities

$$(2.14) \qquad (-1)^{pn} F_q^n F_{pn} = \sum_{k=0}^{n} (-1)^{pk} \binom{n}{k} F_p F_{q-p}^{n-k} F_{qk} \qquad (n = 0, 1, 2, \ldots).$$

If we use the fuller notation $\lambda(p, q)$, $\mu(p, q)$ for λ, μ in (2.13), then,

$$\mu(q, p) = (-1)^{-p-1} \frac{F_p}{F_{q-p}},$$

so that

$$(2.15) \qquad \mu(q, p) = -\lambda(p, q).$$

In proving Theorem 1, we have not made any use of the positivity of p and q. All that is required is that p and q are distinct nonzero integers. This observation gives rise to additional identities. Replacing p by $-p$ in (2.13) we get

$$(2.16) \qquad \lambda(-p, q) = (-1)^p \frac{F_q}{F_{p+q}}, \quad \mu(-p, q) = -\frac{F_p}{F_{p+q}},$$

and (2.14) becomes

$$(2.17) \qquad F_q F_{pn} = -\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_p^k F_{p+q}^{n-k} F_{qk} \qquad (n = 0, 1, 2, \ldots).$$

Comparison of (2.17) with (2.14) yields

$$(2.18)\qquad (-1)^{pn-1}\sum_{k=0}^{n}(-1)^{pk}\binom{n}{k}F_p F_{p+q}^{n-k}F_{qk} = \sum_{k=0}^{n}(-1)^k F_p^k F_{p+q}^{n-k}F_{qk}$$

$$(n = 0,\ 1,\ 2,\ \ldots;\ p^2 \neq q^2).$$

Similarly,

$$(2.19)\qquad \begin{aligned} \lambda(p,\ -q) &= \frac{F_q}{F_{p+q}} = (-1)^p\lambda(-p,\ q),\\ \mu(p,\ -q) &= (-1)^{q-1}\frac{F_p}{F_{p+q}} = (-1)^q\mu(-p,\ q)\end{aligned}$$

and we again get (2.17).

Finally, the formulas

$$(2.20)\qquad \begin{aligned} \lambda(-p,\ -q) &= \frac{F_q}{F_{p+q}} = (-1)^p\lambda(p,\ q),\\ \mu(-p,\ -q) &= (-1)^q\frac{F_p}{F_{p+q}} = (-1)^{p+q}\mu(p,\ q)\end{aligned}$$

again lead to (2.14).

We remark that for $q = p + 1$ and $p + 2$, (2.14) reduces to

$$(2.21)\qquad F_{p+1}^n F_{pn} = \sum_{k=0}^{n}(-1)^{p(n-k)}\binom{n}{k}F_p^k F_{(p+1)k}$$

and

$$(2.22)\qquad F_{p+2}^n F_{pn} = \sum_{k=0}^{n}(-1)^{p(n-k)}\binom{n}{k}F_p^k F_{(p+2)k},$$

respectively.

SECTION 3

We now consider

$$(3.1)\qquad \lambda^n L_{pn} = \sum_{k=0}^{n}\binom{n}{k}\mu^k L_{qn}\qquad (n = 0,\ 1,\ 2,\ \ldots),$$

where p, q are distinct nonzero integers. Since $L_n = a^n + b^n$, we have

$$\lambda^n(a^{pn} + b^{pn}) = \sum_{k=0}^{n}\binom{n}{k}\mu^k(a^{qn} + b^{qn}).$$

Hence,

$$\frac{1}{1-\lambda a^p x} + \frac{1}{1-\lambda b^p x} = \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}\binom{n}{k}\mu^k(a^{qk} + b^{qk})$$

$$= \frac{1}{1-x}\left\{\frac{1}{1-\dfrac{\mu a^q x}{1-x}} + \frac{1}{1-\dfrac{\mu b^q x}{1-x}}\right\},$$

so that

$$\frac{2 - \lambda L_p x}{1 - \lambda L_p x + (-1)^p \lambda^2 x^2} = \frac{2 - (2 + \mu L_q)x}{1 - (2 + \mu L_q)x + \bigl(1 + \mu L_q + (-1)^q \mu^2\bigr)x^2}. \tag{3.2}$$

Equating coefficients and simplifying, we get

$$\begin{cases} \lambda L_p = 2 + \mu L_q \\ (-1)^p \lambda^2 = 1 + \mu L_q + (-1)^q \mu^2. \end{cases} \tag{3.3}$$

Coefficients of x^2 and of x^3 both lead to the second of (3.3).

We can rewrite (3.3) in the form

$$\begin{cases} \lambda(a^p + b^p) = (1 + \mu a^q) + (1 + \mu b^q) \\ \lambda^2 (ab)^p = (1 + \mu a^q)(1 + \mu b^q). \end{cases} \tag{3.4}$$

Squaring the first of (3.4) and subtracting four times the second, we get

$$\lambda^2 (a^p - b^p)^2 = \mu^2 (a^q - b^q)^2,$$

and therefore,

$$\lambda F_p = \pm \mu F_q.$$

If we take $\lambda F_p = \mu L_q$, then, by the first of (3.3),

$$\lambda L_p F_q = 2F_q + \mu L_q F_q = 2F_q + \lambda L_q F_p,$$

that is,

$$\lambda(L_p F_q - L_q F_p) = 2F_q. \tag{3.5}$$

Since, by (2.10),

$$L_p F_q - L_q F_p = 2(-1)^p F_{q-p},$$

we get

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}. \tag{3.6}$$

On the other hand, if $\lambda L_p = -\mu L_q$, then

$$\lambda(L_p F_q + L_q F_p) = 2F_q,$$

which reduces to

$$\lambda = \frac{F_q}{F_{p+q}} = \lambda(p, -q), \quad \mu = -\frac{F_p}{F_{p+q}} = \mu(-p, q). \tag{3.7}$$

This completes the proof of

Theorem 2: Let p, q be fixed nonzero integers, $p \neq q$. Then,

$$\lambda^n L_{pn} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{pk} \qquad (n = 0, 1, 2, \ldots), \tag{3.8}$$

if and only if λ and μ satisfy either (3.6) or (3.7).

Thus we have the explicit identities

$$(-1)^{pn} F_q^n L_{pn} = \sum_{k=0}^{n} (-1)^{pk} \binom{n}{k} F_p^k F_{q-p}^{n-k} L_{qk} \qquad (n = 0, 1, 2, \ldots) \tag{3.9}$$

and

$$F^n L_{pn} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_p^k F_{p+q}^{n-k} L_{qk} \qquad (n = 0, 1, 2, \ldots). \tag{3.10}$$

Note that (3.9) becomes (3.10) if p is replaced by $-p$ or q is replaced by $-q$.

SECTION 4

Let w be a root of $x^2 = x + 1$ and consider

$$(4.1) \qquad \lambda^n w^{pn} = \sum_{k=0}^{n} \mu^k w^{qk} \qquad (n = 0, 1, 2, \ldots),$$

where p, q are fixed nonzero integers, $p \neq q$, and λ and μ are assumed to be rational. Since (4.1) is simply

$$\lambda^n w^{pn} = (1 + \mu w^q)^n \qquad (n = 0, 1, 2, \ldots),$$

it suffices to take $n = 1$:

$$(4.2) \qquad \lambda w^p = 1 + \mu w^q.$$

Recall that

$$(4.3) \qquad w^n = F_n w_n + F_{n-1} \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

so that (4.2) becomes

$$\lambda(F_p w + F_{p-1}) = 1 + \mu(F_q w + F_{q-1}).$$

Since λ and μ are assumed to be rational, we have

$$(4.4) \qquad \begin{cases} \lambda F_p = \mu F_q \\ \lambda F_{p-1} = 1 + \mu F_{q-1}. \end{cases}$$

Eliminating μ, we get

$$\lambda(F_{p-1}F_q - F_p F_{q-1}) = F_q.$$

It is easily verified that

$$F_{p-1}F_q - F_p F_{q-1} = (-1)^p \frac{F_p}{F_{q-p}},$$

and therefore,

$$(4.5) \qquad \lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}.$$

We state

Theorem 3: Let w denote a root of $x^2 = x + 1$ and let p, q be fixed nonzero integers, $p \neq q$. Then,

$$(4.6) \qquad \lambda w^p = 1 + \mu w^q,$$

where p and q are rational, if and only if (4.5) is satisfied. Hence, (4.6) becomes

$$(4.7) \qquad F_q w^p = (-1)^p F_{q-p} + F_p w^q.$$

It follows from (4.7) that

$$F_q^n w^{pn} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F_{q-p}^{n-k} F_p w^{qk},$$

and therefore we get both

$$(4.8) \qquad F_q^n F_{pn} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F_{q-p}^{n-k} F_p^k F_{qk}$$

and

$$(4.9) \qquad F_q^n L_{pn} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F_{q-p}^{n-k} F_p L_{qk},$$

in agreement with (2.14) and (3.9). However, this does not prove Theorems 1 and 2.

SECTION 5

We now discuss

$$(5.1) \qquad \lambda^n F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk+r} \qquad (n = 0, 1, 2, \ldots),$$

where $p \neq q$ but p, q, r are otherwise unrestricted. One would expect that the parameters λ, μ depend on r as well as p and q. However, as will be seen below, λ and μ are in fact independent of r.

It follows from (5.1) that

$$\frac{a^r}{1 - \lambda a^p x} - \frac{b^r}{1 - \mu b^p x} = \frac{1}{1 - x} \left\{ \frac{a^r}{1 - \frac{\mu a^q x}{1 - x}} - \frac{b^r}{1 - \frac{\mu b^q x}{1 - x}} \right\}$$

$$= \frac{a^r}{1 - (1 + \mu a^q)x} - \frac{b^r}{1 - (1 + \mu b^q)x},$$

so that

$$a^r \sum_{k=0}^{\infty} \lambda^k a^{pk} x \; - b^r \sum_{k=0}^{\infty} \lambda^k b^{pk} x^k = a^r \sum_{k=0}^{\infty} (1 + \mu a^q)^k x^k - b^r \sum_{k=0}^{\infty} (1 + \mu b^q)^k x^k.$$

Equating coefficients of x , we get

$$(5.2) \qquad a^r(\lambda^k a^{pk} - (1 + \mu a^q)^k) = b^r(\lambda^k b^{pk} - (1 + \mu b^q)^k)$$

$$(k = 0, 1, 2, \ldots).$$

For $k = 1$, (5.2) implies

$$(5.3) \qquad \lambda F_{p+r} = F_r + \mu F_{q+r}.$$

We now consider separately two possibilities:

(i) $\lambda a^p = 1 + \mu a^q$;
(ii) $\lambda a^p \neq 1 + \mu a^q$.

It is clear from

$$a^r(\lambda a^p - (1 + \mu a^q)) = b^r(\lambda b^p - (1 + \mu b^q))$$

that (i) implies

$$(5.4) \qquad \lambda b^p = 1 + \mu b^q.$$

Subtracting (5.4) from (i), we get

$$(5.5) \qquad \lambda F_p = \mu F_q.$$

Hence, again using (i),

$$(a^p F_q - a^q F_p)\lambda = F_q.$$

Since

$$a^pF_q - a^qF_p = \frac{1}{a - b}(a^p(a^q - b^q) - a^q(a^p - b^p))$$
$$= \frac{a^qb^p - a^pb^q}{a - b} = (-1)^pF_{q-p},$$

it follows that

$$\lambda = (-1)^p\frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p\frac{F_p}{F_{q-p}}. \tag{5.6}$$

We now assume (ii). Take $k = 1, 2, 3$ in (5.2):

$$a^r(\lambda a^p - (1 + \mu a^q)) = b^r(\lambda b^p - (1 + \mu b^q))$$
$$a^r(\lambda^2a^{2p} - (1 + \mu a^q)^2) = b^r(\lambda^2b^{2p} - (1 + \mu b^q)^2)$$
$$a^r(\lambda^3a^{3p} - (1 + \mu a^q)^3) = b^r(\lambda^3b^{3p} - (1 + \mu b^q)^3).$$

Dividing the second and third by the first, we get

$$\begin{gathered}\lambda a^p + (1 + \mu a^q) = \lambda b^p + (1 + \mu b^q)\\ \lambda^2a^{2p} + a^p(1 + \mu a^q) + (1 + \mu a^q) = \lambda^2b^{2p} + \lambda b\ (1 + \mu b^q) + (1 + \mu b^q)^2.\end{gathered} \tag{5.7}$$

The first of (5.7) yields

$$\lambda F_p + \mu F_q = 0 \tag{5.8}$$

while the second gives

$$\lambda^2F_{2p} + \lambda F_p + \lambda\mu F_{p+q} + 2\mu F_q + \mu^2F_{2q} = 0. \tag{5.9}$$

Multiplying (5.9) by F_q and eliminating μ by means of (5.8), we get

$$\lambda^2F_{2p}F_q + \lambda F_pF_q - \lambda^2F_pF_{p+q} - 2\lambda F_pF_q + \lambda^2F_p^2L_q = 0,$$

that is,

$$\lambda(L_pF_q - F_{p+q} + F_pL_q) = F_q.$$

Since

$$L_pF_q - F_{p+q} + F_pL_q = F_{p+q},$$

we have, finally,

$$\lambda = \frac{F_q}{F_{p+q}}, \quad \mu = -\frac{F_p}{F_{p+q}}. \tag{5.10}$$

On the other hand, it follows from (5.3) and (5.8) that

$$\lambda(F_{p+r}F_q + F_{q+r}F_p) = F_rF_q.$$

This gives

$$\lambda = \frac{F_rF_p}{F_{p+r}F_q + F_{q+r}F_p} \neq \frac{F_q}{F_{p+q}}.$$

Hence, possibility (ii) is untenable and only the value of λ and μ furnished by (5.6) need be considered.

Conversely, since (5.6) implies $\lambda a^p - (1 + \mu a^q) = 0 = \lambda b^p - (1 + \mu b^q)$, and this in turn implies (5.2), it is clear that (5.1) holds only if (5.6) is satisfied.

This completes the proof of the following

Theorem 4: Let p, q be fixed nonzero integers, $p \neq q$, and let r be an arbitrary integer. Then,

$$\lambda^n F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k F_{qk+r} \qquad (n = 0, 1, 2, \ldots), \tag{5.11}$$

if and only if

$$\lambda = (-1)^p \frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p \frac{F_p}{F_{q-p}}. \tag{5.12}$$

Thus, we have the explicit identity

$$F_q^n F_{pn+r} = \sum_{k=0}^{n} (-1)^{p(n-k)} \binom{n}{k} F_p^k F_{q-p}^{n-k} F_{qk+r} \qquad (n = 0, 1, 2, \ldots). \tag{5.13}$$

We note that, as stated, (5.13) holds for arbitrary integers p, q, r. In particular, for $q = -p$, (5.13) becomes

$$F_p^n F_{pn+r} = \sum_{k=0}^{n} (-1)^{(p-1)(n-k)} \binom{n}{k} F_p^k F_{2p}^{n-k} F_{-pk+r} \qquad (n = 0, 1, 2, \ldots). \tag{5.14}$$

SECTION 6

We turn finally to

$$\lambda^n L_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} \mu^k L_{qk+r} \qquad (n = 0, 1, 2, \ldots). \tag{6.1}$$

It follows from (6.1) that

$$\frac{a^r}{1 - \lambda a^p x} + \frac{b^r}{1 - \lambda b^p x} = \frac{a^r}{1 - (1 + \mu a^q)x} + \frac{b^r}{1 - (1 + \mu b^q)x}.$$

Hence,

$$a^r(\lambda^k a^{pk} - (1 + \mu a^q)^k) = -b^r(\lambda^k b^{pk} - (1 + \mu b^q)^k) \quad (k = 0, 1, 2, \ldots). \tag{6.2}$$

For $k = 1$, (6.2) implies

$$\lambda L_{p+r} = L_r + \mu L_{q+r}. \tag{6.3}$$

As in §5, we again consider the two possibilities:

(i) $\lambda a^p = 1 + \mu a^q$;
(ii) $\lambda a^p \neq 1 + \mu a^q$.

It is clear from

$$a^r(\lambda a^p - (1 + \mu a^q)) + b^r(\lambda b^p - (1 + \mu b^q)) = 0$$

and (i) that

$$\lambda b^p = 1 + \mu b^q. \tag{6.4}$$

Adding together (i) and (6.4), we get

$$\lambda L_p = 2 + \mu L_q. \tag{6.5}$$

Again using (i),

$$\lambda(a^qL_p - a^pL_q) = a^q - b^q,$$

which gives

$$\lambda = (-1)^p\frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p\frac{F_p}{F_{q-p}}, \tag{6.6}$$

Assuming (ii), we have

$$a^r(\lambda a^p - (1 + \mu a^q)) = -b^r(\lambda b^p - (1 + \mu b^q))$$
$$a^r(\lambda^2 a^{2p} - (1 + \mu a^q)^2) = -b^r(\lambda^2 b^{2p} - (1 + \mu b^q)^2)$$
$$a^r(\lambda^3 a^{3p} - (1 + \mu a^q)^3) = -b^r(\lambda^3 b^{3p} - (1 + \mu b^q)^3).$$

This gives

$$\begin{cases} \lambda a^p + (1 + \mu a^q) = \lambda b^p + (1 + \mu b^q) \\ \lambda^2 a^{2p} + \lambda a^p(1 + \mu a^q) + (1 + \mu a^q)^2 = \lambda^2 b^{2p} + \lambda b^p(1 + \mu b^q) + (1 + \mu b^q)^2. \end{cases} \tag{6.7}$$

Then, exactly as in the previous section, we get

$$\lambda = \frac{F_q}{F_{p+q}}, \quad \mu = -\frac{F_p}{F_{p+q}}. \tag{6.8}$$

On the other hand, by (6.3) and the first of (6.7), that is,

$$\lambda F_p = \mu F_q = 0,$$

we get

$$\lambda(F_qL_{p+r} + F_pL_{q+r}) = L_qL_r.$$

This gives

$$\lambda = \frac{L_qL_r}{F_qL_{p+r} + F_pL_{q+r}} \neq \frac{F_q}{F_{p+q}} \qquad (r \neq 0).$$

Hence, (ii) leads to a contradiction and only (i) need be considered. Since (6.6) implies (i), it is clear that (6.1) holds only if (6.6) is satisfied.

We may state

Theorem 5: Let p, q, r be fixed nonzero integers, $p \neq q$, $r \neq 0$. Then we have

$$\lambda^n L_{pn+r} = \sum_{k=0}^{n}\binom{n}{k}\mu^k L_{qn+r} \qquad (n = 0, 1, 2, \ldots) \tag{6.9}$$

if and only if

$$\lambda = (-1)^p\frac{F_q}{F_{q-p}}, \quad \mu = (-1)^p\frac{F_p}{F_{q-p}}. \tag{6.10}$$

Thus, we have

$$F_q^n L_{pn+r} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F_p^kF_{q-p}^{n-k}L_{qn+r} \qquad (n = 0, 1, 2, \ldots) \tag{6.11}$$

for all p, q, r.

Remark: Theorem 5 does not include Theorem 2 since, for $r = 0$, λ, μ may also take on the values (3.7).

SECTION 7

The identity

$$L_{pn} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_p^{n-k} L_{pk} \qquad (n = 0, 1, 2, \ldots) \tag{7.1}$$

has been noted in the Introduction. This suggests the problem of finding sequences $U = \{u_0, u_1, u_2, \ldots\}$ such that

$$u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_1^{n-k} u_k \qquad (n = 0, 1, 2, \ldots). \tag{7.2}$$

The sequence U is not uniquely determined by (7.2). We shall assume that $u_1 \neq 0$. For $n = 1$, we have $u_1 = u_0u_1 - u_1$, so that $u_0 = 2$. For $n = 2$, we get $u_2 = u_0u_1^2 - 2u_1^2 + u_2$. For $n = 2m$, $m > 0$, (7.2) reduces to

$$\sum_{k=0}^{2m-1} (-1)^k \binom{2m}{k} u_1^{2m-k} u_k = 0 \qquad (m = 1, 2, 3, \ldots). \tag{7.3}$$

For $n = 2m - 1$, (7.2) yields

$$2u_{2m-1} = \sum_{k=0}^{2m-2} (-1)^k \binom{2m-1}{k} u_1^{2m-k-1} u_k \qquad (m = 1, 2, 3, \ldots). \tag{7.4}$$

Put

$$S_n \equiv \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_1^{n-k} u_k.$$

Then

$$u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_1^{n-k} S_k,$$

so that

$$u_n - S_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_1^{n-k} (S_k - u_k)$$

and so

$$-2(S_{2m} - u_{2m}) = \sum_{k=0}^{2m-1} (-1)^k \binom{2m}{k} u_1^{2m-k} (S_k - u_k). \tag{7.5}$$

Hence (7.4) is a consequence of the earlier relations

$$S_k = u_k \qquad (k = 1, 2, 3, \ldots, 2m - 2).$$

In the next place, if we put

$$G(x) = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!},$$

it follows from (7.2) that

$$G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_1^{n-k} u_k = \sum_{n=0}^{\infty} (-1)^k u_k \frac{x^k}{k!} \sum_{k=0}^{n} \frac{(u_1x)^n}{n!}.$$

Thus,

$$G(x) = e^{u_1 x} G(-x).$$

In particular, the sequence $\{L_0, L_p, L_{2p}, \ldots\}$, with $u_1 = L_p$, satisfies (7.2); incidentally, a direct proof of (7.1) is easy. Hence, if we put

$$G_L(x) = \sum_{n=0}^{\infty} L_{pn} \frac{x^n}{n!},$$

we have

(7.6) $$G_L(x) = e^{u_1 x} G_L(-x) \qquad (u_1 = L_p).$$

It then follows from (7.6) that

$$F(x) = G(x)/G_L(x) = F(-x).$$

Thus,

$$F(x) = \sum_{k=0}^{\infty} c_2 \frac{x^{2k}}{(2k)!} \qquad (c_0 = 1),$$

where the coefficients c_2, c_4, c_6, ... are arbitrary. We have therefore,

(7.7) $$u_n = \sum_{2k \le n} \binom{n}{2k} c_{2k} L_{p(n-2)}$$

for any sequence satisfying (7.2) with $a_1 = L_p$.

This result also suggests a method for handling (7.2) when u_n is arbitrary. Put

(7.8) $$u_1 = \alpha + \beta,$$

where α, β are unrestricted otherwise. Then we have

$$\begin{aligned}
&\sum_{k=0}^{n} (-1)^k \binom{n}{k} (\alpha + \beta)^{n-k} (\alpha^k + \beta^k) \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha^k \sum_{j=0}^{n-k} \binom{n-k}{j} \alpha^{n-k-j} \beta^j + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \beta^k \sum_{j=0}^{n-k} \binom{n-k}{j} \alpha^{n-k-j} \beta^j \\
&= \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \beta^j \sum_{k=0}^{n-k} (-1)^k \binom{n-j}{k} + \sum_{s=0}^{n} \binom{n}{s} \alpha^{n-s} \beta^s \sum_{k=0}^{s} (-1) \binom{s}{k} = \alpha^n + \beta^n.
\end{aligned}$$

Hence, if we define

(7.9) $$u_n = \alpha^n + \beta^n \qquad (n = 0, 1, 2, \ldots),$$

it is clear that

(7.10) $$u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_1^{n-k} u_k \qquad (n = 0, 1, 2, \ldots).$$

Thus (7.2) is satisfied with u_n defined by (7.9).

We can now complete the proof of the following theorem exactly as for the special case $u_1 = L_p$.

Theorem 6: The sequence $\{u_0 = 2, u_1, u_2, \ldots\}$ satisfies (7.10) if and only if

(7.11) $$u_n = \sum_{2k \le n} \binom{n}{2k} c_{2k} u_{n-2k} \qquad (n = 0, 1, 2, \ldots),$$

where $c_0 = 1$ and c_2, c_4, c_6, ... are arbitrary. An equivalent criterion is

(7.12) $$u_n = \alpha^n + \beta^n \qquad (n = 0, 1, 2, \ldots)$$

for some fixed α, β.

We remark that

$$\alpha^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+\beta)^{n-k}\alpha^k \qquad (n = 1, 2, 3, \ldots)$$

is not correct. For example,

$$(\alpha+\beta) - \alpha = \beta$$
$$(\alpha+\beta)^2 - 2(\alpha+\beta)\alpha + \alpha^2 = \beta^2.$$

We shall prove

$$(7.13)\qquad \left\{\begin{aligned} \beta^n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+\beta)^{n-k}\alpha^k \\ \alpha^n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+\beta)^{n-k}\beta^k \end{aligned}\right. \qquad (n = 0, 1, 2, \ldots).$$

It suffices to prove the first of (7.13). We have

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+\beta)^{n-k}\alpha^k &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\alpha^k\sum_{j=0}^{n-k}\binom{n-k}{j}\alpha^{n-k-j}\beta^j \\ &= \sum_{j=0}^{n}\binom{n}{j}\alpha^{n-j}\beta^j\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k} = \beta^n.\end{aligned}$$

This completes the proof. Note that this result had occurred implicitly in the discussion preceding (7.10).

It follows from (7.13) after multiplication by α^r (or β^r) that

$$(7.14)\qquad (\alpha\beta)^r u_{n-r} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}u_{k+r} \qquad (n = 0, 1, 2, \ldots),$$

where now $u_n = \alpha^n + \beta^n$ for all integral n. Similarly, we have

$$(7.15)\qquad -(\alpha\beta)^r v_{n-r} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}u_1^{n-k}v_{k+r} \qquad (n = 0, 1, 2, \ldots),$$

where

$$(7.16)\qquad v_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

In both (7.14) and (7.16), r is an arbitrary integer.

In the case of the Lucas and Fibonacci numbers, we can improve slightly on (7.14) and (7.16) by first taking $\alpha = a^p$, $\beta = b^p$ in (7.13) and then multiplying by a^r (or b^r). Thus, we get

$$(7.17)\qquad (-1)^r L_{pn-r} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}L^{n-k}L_{pk+r} \qquad (n = 0, 1, 2, \ldots)$$

and

$$(7.18)\qquad (-1)^{r-1}F_{pn-r} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}L^{n-k}F_{pk+r} \qquad (n = 0, 1, 2, \ldots),$$

where r is an arbitrary integer.

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FIBONACCI CHROMOTOLOGY OR HOW TO PAINT YOUR RABBIT

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Readers of this journal are aware that Fibonacci numbers have been used to generate musical compositions [1], [2], and that the Golden Section ratio has appeared repeatedly in art and architecture. However, that Fibonacci numbers can be used to select colors in planning a painting is less well-known and certainly an exciting application.

One proceeds as follows, using a color wheel based upon the color theory of Johann Wolfgang von Goethe (1749-1832) and developed and extended by Fritz Faiss [3]. Construct a 24-color wheel by dividing a circle into 24 equal parts as in Figure 1. Let 1, 7, 13, and 19 be yellow, red, blue, and green, respectively. (In this system, green is both a primary color and a secondary color.) Halfway between yellow and red, place orange at 4, violet at 10, blue-green at 16, and yellow-green at 22. The other colors must proceed by even graduations of hue. For example, 2 and 3 are both a yellow-orange, but 2 is a yellow-yellow-orange, while 3 is a more orange shade of yellow-orange. The closest colors to use are: (You must also use your eye.)

1	Cadmium Yellow Light
2	Cadmium Yellow Medium
3	Cadmium Yellow Deep
4	Cadmium Orange or Vermilion Orange
5	Cadmium Red Light or Vermilion
6	Cadmium Red Medium
7	Cadmium Red Deep or Acra Red
8	Alizarin Crimson Golden or Acra Crimson
9	Rose Madder or Alizarin Crimson
10	Thalo Violet or Acra Violet
11	Cobalt Violet
12	Ultramarine Violet or Permanent Mauve or Dioxine Purple
13	Ultramarine Blue
14	French Ultramarine or Cobalt Blue
15	Prussian Blue
16	Thalo Blue or Phthalocyanine Blue or Cerulean Blue or Manganese Blue
17	Thalo Blue + Thalo Green
18	Thalo Green + Thalo Blue
19	Thalo Green or Phthalocyanine Green
20	Viridian